Some exotic tensor categories in prime characteristic

(Joint work with Etingof and Ostrik)

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Goal

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Construct some new symmetric tensor categories Ver_{p^n} in characteristic p, and explain some of their properties.

Definition

k algebraically closed, char p. $\mathcal{C}$ a rigid symmetric tensor abelian category over k:

- Bilinear tensor product $X \otimes Y$ exact
- Tensor identity $\mathbb{1}$
- Symmetric braiding $X \otimes Y \xrightarrow{\cong} Y \otimes X$ of order 2
- Rigid: objects have duals ...
- Coherence conditions, so for example $k\Sigma_n$ acts on $X^{\otimes n}$

Moderate growth

Definition

X in $\mathcal{C}$ has moderate growth if $\exists$ Schur functor S^λ such that $S^\lambda X = 0$.

Remark

The tensor nth power functor has a filtration by Schur functors, whose length grows worse than exponentially with n. So if the length of $X^{\otimes n}$ is finite and grows at worst exponentially with n then X has moderate growth.

Definition

We say $\mathcal{C}$ has moderate growth if every object in $\mathcal{C}$ does.

Deligne's Theorem

Theorem (Deligne)

If $\mathcal{C}$ has moderate growth in characteristic zero then $\exists$ symmetric tensor functor $\mathcal{C} \to \mathrm{SVec}$ that doesn't kill any object.

Such a functor to a simpler category that we think we understand is called a fibre functor.

Super vector spaces: $V = V_0 \oplus V_1$, $|v| = 0$ or 1

Symmetric braiding: $x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$

Tannakian consequence: $\mathcal{C} \simeq$ f.d. reps of (pro-)affine supergroup scheme (plus a bit more structure), constructed as $\mathrm{Aut}(\mathcal{C} \to \mathrm{SVec})$.

Point of view: The only incompressible symmetric tensor categories of moderate growth in characteristic zero are vector spaces and super vector spaces.

What about characteristic p?

Definition

A symmetric tensor category $\mathcal{C}$ is finite if $\simeq$ Rep(f.d. alg).

Remark

Rigidity implies that the f.d. algebra is automatically self-injective.

Theorem (Ostrik)

If $\mathcal{C}$ is finite and semisimple then $\exists$ fibre functor $\mathcal{C} \to \mathsf{Ver}_p$.

Consequence: $\mathcal{C} \simeq$ f.d. reps of semisimple finite group scheme over Ver_p.

What is Ver_p?

Take $\mathrm{Rep}(\mathbb{Z}/p)$ and semisimplify: quotient out all $f\colon X \to Y$ that have the property: $\forall g\colon Y \to X$ we have $\mathrm{Tr}(fg) = 0$ in k.
Objects: $S_1, \dots, S_{p-1}$ corresp to Jordan blocks of length $1, \dots, p-1$

$$S_2 \otimes S_i \cong S_{i-1} \oplus S_{i+1}$$

(Convention: $S_0 = S_p = 0$)

Remark

$\mathrm{Ver}_p \simeq \mathrm{Ver}_p^+ \boxtimes \mathrm{SVec}(k)$ where Ver_p^+ has simples $S_1, S_3, \dots, S_{p-2}$.

e.g.
$\mathrm{Ver}_2 = \mathrm{Vec}$
$\mathrm{Ver}_3 = \mathrm{SVec}$
$\mathrm{Ver}_5 = \mathrm{Ver}_5^+ \boxtimes \mathrm{SVec}$
Ver_5^+ has two simples $S_1 = \mathbb{1}$, S_3 with $S_3 \otimes S_3 \cong S_3 \oplus \mathbb{1}$.
So the "dimension" of S_3 is the golden ratio $(1 + \sqrt{5})/2$.

What about $\mathcal{C}$ non-semisimple?

Example (Venkatesh 2016)

$\mathrm{char}(k) = 2$

Objects in $\mathcal{C}$ are pairs (V, d) where $d\colon V \to V$ with $d^2 = 0$

Tensor products: $d(x \otimes y) = dx \otimes y + x \otimes dy$

Symmetric braiding: $x \otimes y \mapsto y \otimes x + dy \otimes dx$.

This example is incompressible.

Remark

This example appears in mod 2 Morava K-theories.

Char 2: B-Etingof (Advances, 2019)
Odd char: B-Etingof-Ostrik, and independently Kevin Coulembier

We construct incompressible symmetric tensor categories Ver_{p^n} in characteristic p with properties:

- Finite ($\simeq$ Mod(f.d. alg))
- For $p = 2$, $\mathsf{Vec} = \mathsf{Ver}_2 \subseteq \mathsf{Ver}^+_{2^2} \subseteq \mathsf{Ver}_{2^2} \subseteq \mathsf{Ver}^+_{2^3} \subseteq \cdots$
- For p odd, $\mathsf{Ver}_{p^n} = \mathsf{Ver}^+_{p^n} \boxtimes \mathsf{SVec}$, and two chains
  $\mathsf{SVec} \subseteq \mathsf{Ver}_p \subseteq \mathsf{Ver}_{p^2} \subseteq \cdots$
  $\mathsf{Vec} \subseteq \mathsf{Ver}^+_p \subseteq \mathsf{Ver}^+_{p^2} \subseteq \cdots$
- For $n \geqslant 2$, Ver_{p^n} is not semisimple
- Ver_{p^n} has $p^{n-1}(p-1)$ iso classes of simples
- The Grothendieck ring of $\mathsf{Ver}^+_{p^n}$ is $\mathbb{Z}[2\cos(\pi/p^n)]$

Conjecture

If $\mathcal{C}$ is a finite symmetric tensor category over an algebraically closed field k of characteristic p then there exists a fibre functor $\mathcal{C} \to \mathsf{Ver}_{p^n}$ for some $n \geq 0$.

Let $\mathsf{Ver} = \varinjlim \mathsf{Ver}_{p^n}$, an infinite incompressible symmetric tensor category.

Conjecture

If $\mathcal{C}$ is a symmetric tensor category of moderate growth over an algebraically closed field k of characteristic p then there exists a fibre functor $\mathcal{C} \to \mathsf{Ver}$.

Tilting modules for $SL(2, k)$

k alg closed of char p, $SL(2, k)$ alg group of 2x2 matrices of det 1.

Inductive defn: $T_0 = k$, $T_1 = V$, natural 2-dim module

$T_{n-1} \otimes V$ has unique indec summand not isomorphic to any T_i, $i < n$. This is T_n.

Thus $V^{\otimes n} = T_n \oplus$ smaller.

e.g. for $n < p$, $T_n = S^n(V)$ of dimension $n + 1$

T_{p-1} is first Steinberg module

T_p is uniserial, composition factors $S^{p-2}(V)$, $F(V)$, $S^{p-2}(V)$

$\mathrm{St}_n = T_{p^n-1}$ is the Steinberg module of dimension p^n.

Fact

If $j \geq p^n - 1$, $T_i \otimes T_j$ is a sum of T_ℓs with $\ell \geq p^n - 1$, so $\{T_j,\ j \geq p^n - 1\}$ is a tensor ideal.

$\mathrm{St}_n = T_{p^n-1}$ is the Steinberg module of dimension p^n.

Fact

If $j \geq p^n - 1$, $T_i \otimes T_j$ is a sum of T_ℓs with $\ell \geq p^n - 1$, so $\{T_j,\ j \geq p^n - 1\}$ is a tensor ideal.

Proposition

Let X and Y be in $\mathrm{Add}\Big(\bigoplus_{i=0}^{p^n-2} T_i\Big)$ and $f\colon X \to Y$. Then $f \otimes 1\colon X \otimes \mathrm{St}_{n-1} \to Y \otimes \mathrm{St}_{n-1}$ splits (i.e., direct sum of an isomorphism and a zero map).

Proof: not very hard.

DEFINITION

$$T = \bigoplus_{i=p^{n-1}-1}^{p^n-2} T_i, \qquad A = \mathrm{End}(T)^{\mathrm{op}}, \qquad \mathrm{Ver}_{p^n} = \mathrm{mod}(A).$$

Projective indecomposable A-modules $P_j = \mathrm{Hom}_{SL(2,k)}(T, T_j)$

Symmetric tensor structure on projectives inherited from tensors over $SL(2,k)$

THEOREM

Let M, N be f.g. A-modules, with projective resolutions $P_* \to M$, $Q_* \to N$. Then $P_* \otimes Q_*$ is exact except in degree zero.

So define $M \otimes N = H_0(P_* \otimes Q_*)$, check well defined, exact, etc.

Some cohomology rings, $p = 2$

k a field of characteristic 2
Take a $\mathbb{Z}[\frac{1}{2}]$-graded polynomial algebra

generator	x_1	x_2	x_3	$\cdots$	x_n
degree	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{7}{8}$	$\cdots$	$\frac{2^n-1}{2^n}$

$\mathcal{E}_n(2) =$ elements of integer degree in $k[x_1, \dots, x_n]$

e.g. $\mathcal{E}_2(2) = k[u, v, w]/(uw - v^2)$, $\quad u = x_1^2$, $v = x_1x_2^2$, $w = x_2^4$

$\mathcal{E}_n(2)$ is a Gorenstein integral domain, with hsop $x_1^2, x_2^4, \dots, x_n^{2^n}$.

Conjecture

$\mathrm{Ext}^*_{\mathsf{Ver}_{2^{n+1}}}(\mathbb{1}, \mathbb{1}) \cong \mathcal{E}_n(2)$.

Some cohomology rings, p odd

k a field of characteristic p odd

$\mathcal{E}_n(p) =$ elements of integer degree in $k[x_1, \dots, x_n] \otimes \Lambda(y_1, \dots, y_n)$

$\deg(x_i) = 2(p^i - 1)/p^i$, and $\deg(y_i)$ is one less, $(p^i - 2)/p^i$.

$\mathcal{E}_n(p)$ is a Gorenstein ring with hsop $x_1^p, x_2^{p^2}, \dots, x_n^{p^n}$.

Conjecture

$\mathrm{Ext}^*_{\mathsf{Ver}_{p^{n+1}}}(\mathbb{1}, \mathbb{1}) \cong \mathcal{E}_n(p)$.

Minc's partition function

($p=2$) Let $\mathcal{E}_\infty(2) = \varinjlim \mathcal{E}_n(2)$, conjecturally $\mathrm{Ext}^*_{\mathsf{Ver}}(\mathbb{1},\mathbb{1})$.
Poincaré series $f(q) = \sum_{i\ge 0} q^i \dim(\mathcal{E}_\infty(2))_i$ is
Minc's partition function - investigated by Andrews.
He proved a Rogers-Ramanujan type formula for $1/f(q)$:

$$1 - \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^3)} - \frac{q^{11}}{(1-q)(1-q^3)(1-q^7)} + \cdots$$

(p odd) We have a similar formula, more complicated.
See our paper arXiv:2008.13149.