Representations for k-graph C^*-algebras

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Joint on-going collaboration with E. Gillaspy, P. Jorgensen, S. Kang and J. Packer; also joint with D. Gonçalves
ICMS 2019 Edinburgh
All k-graphs in this talk are row-finite, strongly connected and source free. In many cases, they are also finite, and $k = 2$

Motivation

k-Graphs from an analyst's viewpoint

- $k = 1$, 1-Graphs
- k-Graphs (here $k = 2$)

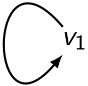
Representations: Faithful, Monic and Atomic

Construction of Representations: Λ-Semibranching Function Systems

- Semibranching Function System (sbfs) for 1-graphs
- Λ-Semibranching Function Systems
- The Standard Representation on the Infinite Path Space

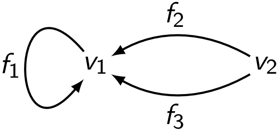
Why k-graphs from an analyst's viewpoint?

Because they provide means of constructing examples of interesting C^*-algebras. C^*-algebras are particular algebras of operators with an involution and satisfying the special C^*-identity.
The graph below given by a single loop give rise to the C^*-algebra $C(\mathbb{T})$ with involution $f^*(z) = \overline{f(\overline{z})}$ (but C^*-algebras associated to many other graphs are not so easy to determine!).



$$C^* - algebra = C(\mathbb{T})$$

What is a 1-graph (or graph)? Definitions and Examples



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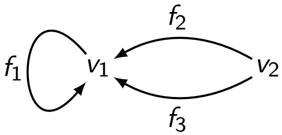

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  - Finitely many points (vertices);

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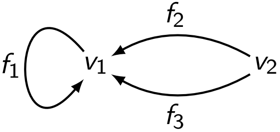

- A (skeleton) graph consists of
  - Finitely many points (vertices);and
  - Finitely many segments connecting some of the vertices (edges).

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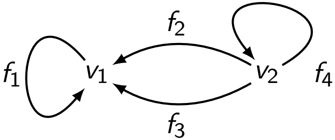
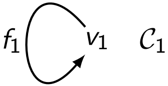


- vertex incidence matrix $M = (m_{i,j})$ of size $n \times n$, with $n =$ #(vertices).
- Entry $m_{i,j} =$ #(edges from j to i).
- $$M = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

Examples of 1-graph: $\mathcal{C}_1$

- Graph with one vertex M is 1×1;
- entry $= \#$(loops).



Vertex Matrix: $M = \begin{pmatrix} 1 \end{pmatrix}$

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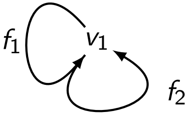

$\mathcal{C}_2$

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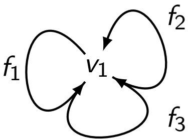

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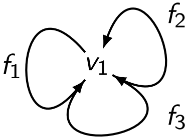

$\mathcal{C}_3$

Vertex Matrix: $A = \begin{pmatrix} 3 \end{pmatrix}$

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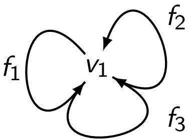

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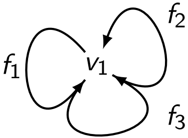

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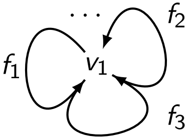

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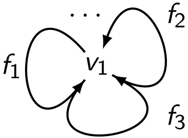

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Vertex Matrix: $A = \left(n\right)$

Example of 1-graph: $\mathcal{C}_n$

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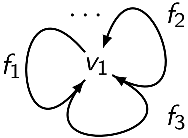

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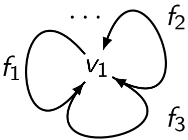
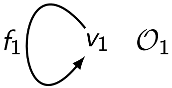


$\mathcal{C}_n$

Vertex Matrix: $A = (n)$

Finite Paths in a 1-graph

A finite path space in a graph is a finite string of connecting edges. In the $\mathcal{O}_1$ example:



Vertex matrix

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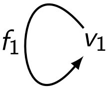

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Vertex matrix

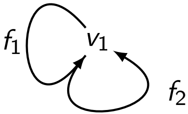
$$M = (1).$$

Examples of finite paths in $\mathcal{C}_1$:

$$\alpha_n = (f_1)^n.$$

Finite Paths in a graph

A finite path space in a graph is a finite string of connecting edges. In the $\mathcal{C}_2$ example:



$\mathcal{O}_2$

Vertex matrix

$$M = (2).$$

Finite Paths in a graph

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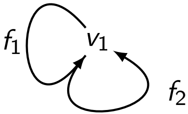
In the $\mathcal{C}_2$ example:



$\mathcal{O}_2$

Vertex matrix

$$M = (2).$$

Examples of finite paths in $\mathcal{C}_2$:

$$\alpha = f_1 f_1;\ \beta = f_1 f_2;\ \gamma = f_2 f_2 f_2,\ \delta = f_1 f_2 f_1 f_1.$$

Finite Paths:

$$v_1 \xleftarrow[f_1]{} v_1 \xleftarrow[f_2]{} v_1$$

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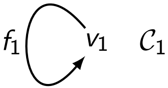
the Infinite Path Space of a 1-graph

- An infinite path space in a graph is an infinite string of connecting edges.
- The **Infinite Path Space** of a graph is the set of infinite paths of the graph.



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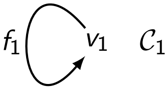

Finite paths: $(f_1)^n$, with $n \in \mathbb{N}$. Only one Infinite path:

$$\alpha = (f_1)^\infty.$$

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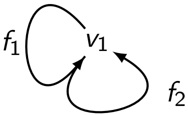

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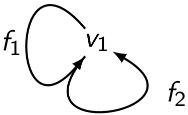


$\mathcal{C}_2$

Infinite paths:

$$\alpha = f_1 f_2 \ldots;\ \beta = f_1 f_1 f_2 \ldots;\ \gamma = f_1 f_2 f_1 f_1 \ldots.$$

Infinite Paths:

$$\alpha : \qquad v_1 \xleftarrow[f_1]{} v_1 \xleftarrow[f_2]{} v_1 \quad \cdots$$

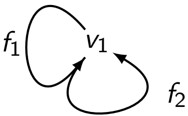
$$\beta : \qquad v_1 \xleftarrow[f_1]{} v_1 \xleftarrow[f_1]{} v_1 \xleftarrow[f_2]{} v_1 \quad \cdots$$

Example: the Infinite Path Space of $\mathcal{C}_2$

In this case any infinite string of f_1 and f_2 (in any combination) is an infinite path:

$$f_{i_1} f_{i_2} f_{i_3} f_{i_4} f_{i_5} f_{i_6} \dots,$$

with $i_j \in \{0, 1\}$.



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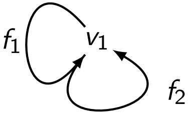


$\mathcal{C}_2$

The Infinite Path Space is the set of the infinite Paths of the graph.

Finite Path and associated cylinder set:

$$v_1 \xleftarrow[f_{i_1}]{} v_1 \xleftarrow[f_{i_2}]{} v_1 \xleftarrow[f_{i_3}]{} v_1 \cdots\cdot$$

Perron-Frobenius Measure on the Infinite Path Space

Defined on cylinder sets by

Perron-Frobenius Measure on the Infinite Path Space

Defined on cylinder sets by

- Assume the vertex matrix M is irreducible. Then it admits a unique positive eigenvector of ℓ^1-norm 1

  $$(x_1, \ldots, x_1) \text{ (where } n \text{ is the number of vertices)}$$

  with eigenvalue the spectral radius $\rho(M)$ of M.
- Then

  $$measure(Z(f_{i_1} f_{i_2} \ldots f_{i_s})) = \rho(M)^{-s} x_{s(f_{i_1} f_{i_2} \ldots f_{i_s})}$$

  with $i_j \in \{0, 1\}$,
- By Carathéodory-Kolmogorov, this extends to the σ-algebra generated by the cylinder sets.
- Thus the Infinite Path Space of 1-graphs is endowed with the Perron-Frobenius measure.

Example: the Infinite Path Space of $\mathcal{C}_2$

- To any infinite string

$$f_{i_1} f_{i_2} f_{i_3} f_{i_4} f_{i_1 5} f_{i_6} \ldots,$$

with $i_j \in \{0, 1\}$,

- we can bijectively associate

$$f_{i_1} f_{i_2} f_{i_3} f_{i_4} f_{i_5} f_{i_6} \ldots \iff i_1 i_2 i_3 i_4 i_5 i_6 \ldots$$

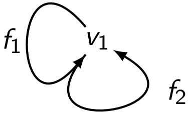
a sequence of 0 and 1.

- So the Infinite Path Space is in this example

$$\prod_{n \in \mathbb{N}} \{0, 1\},$$

the Cantor set.

Example: the measure on the Infinite Path Space of $\mathcal{C}_2$



$\mathcal{O}_2$

- $$M = (n),\ \rho(A) = n,\ x = (1) \text{ Perron-Frobenius eigenvetor}$$

- $$\text{measure of}\Big(Z(f_{i_1} \dots f_{i_s})\Big) = (n)^{-s}\, 1$$

The $\mathcal{C}_2$ example

- Cylinder sets: any finite path we define its associated cylinder set by

$$Z(f_{i_1} f_{i_2} \dots f_{i_k}) = \{ \text{ All infinite paths starting with } f_{i_1} f_{i_2} \dots f_{i_k} \}.$$

- Cylinder sets generate a topology and a Borel structure on the infinite path space of the 1-graph

- What is $k = 2$-**graph**?

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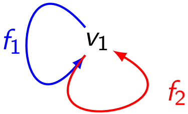
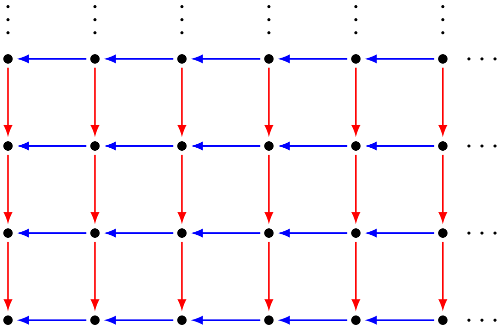
- The idea is to take a **2-dimensional analog** of a graph, with the addition of some data (factorisation rules).



A finite path in a 2-Graph

- For $k = 2$, Finite and Infinite Paths can be drawn on the nonnegative quadrant of $\mathbb{N} \times \mathbb{N}$.



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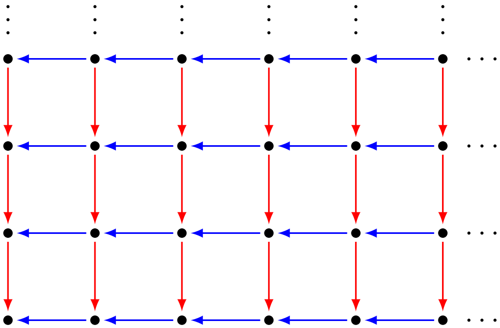

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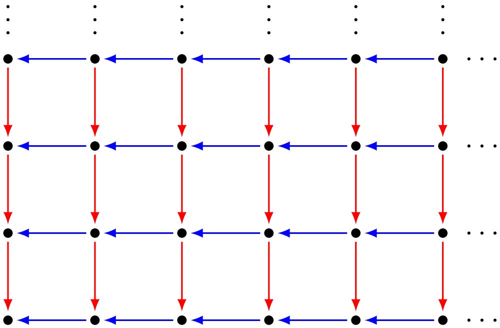
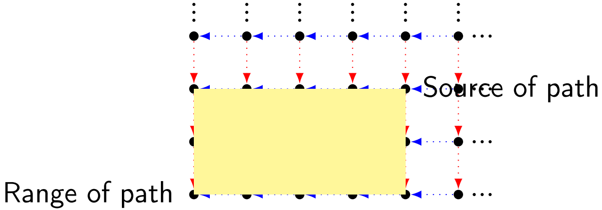


Edges (of color 1–blue) have degree $(1, 0)$
Edges (of color 2–red) have degree $(0, 1)$.

A Finite Path of Degree =(4,2):



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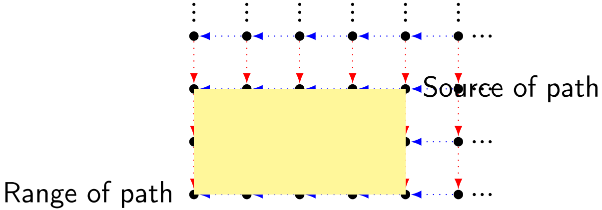

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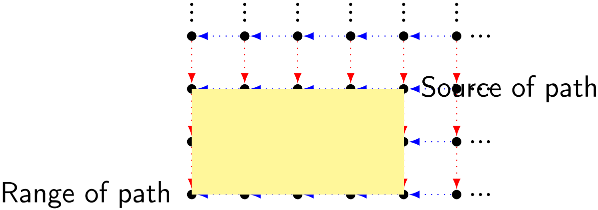

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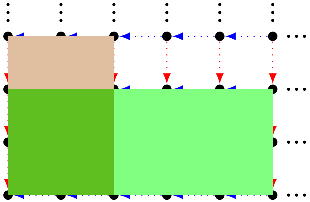
Finite Paths in 2-Graphs and their Degree

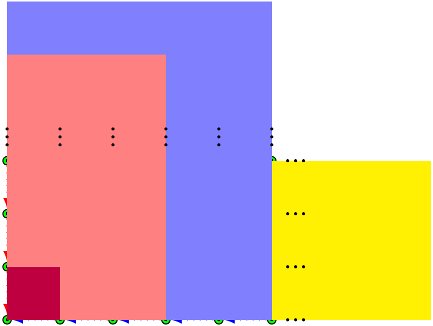
Two Finite Paths and their Degree:



Degree =(5,2), Degree =(2,3)

Cylinder Sets

Cylinder Sets generate the topology and the Borel structure of Λ^∞:



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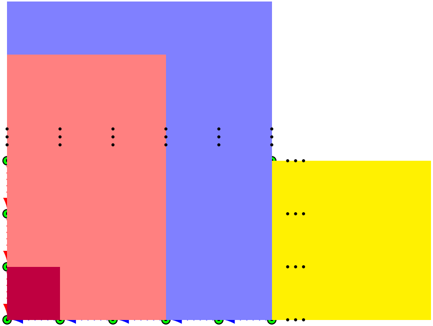

The measure is the Hausdorff measure.

Measure on the Infinite Path Space of a 2-Graph (AnHuef-Laca-Raeburn-Sims)

- When Λ is strongly connected the vertex matrices A_1, A_2 have a common positive Perron-Frobenous eigenvector of norm 1: $x = (x_1, \ldots, x_n)$ with eigenvalues the spectral radii $\rho(A_1), \rho(A_2)$ of A_1, A_2

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- Define $\rho(\Lambda) := \rho(A_1)\rho(A_2)$.

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- Define $\rho(\Lambda) := \rho(A_1)\rho(A_2)$.
- The Perron-Frobenius measure M of a the cylinder set associated to a finite path is

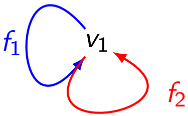
$$M(Z(\lambda)) := \rho(\Lambda)^{-deg\lambda} x_{s(\lambda)}$$

$$M(Z(\lambda)) := \rho(A_1)^{deg(\lambda_1)} \rho(A_2)^{deg(\lambda_2)} x_{s(\lambda)}$$

- By Carathódory-Kolmogorov, this measure can be extended to the Borel sets on the Infinite Path Space.

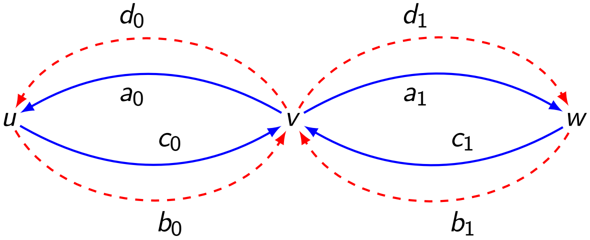
Examples of 2-graphs

This is an example.

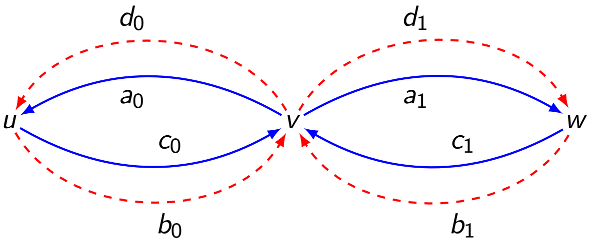


Factorization Relations: $f_1 f_2 = f_2 f_1$.

Example of 2-graph

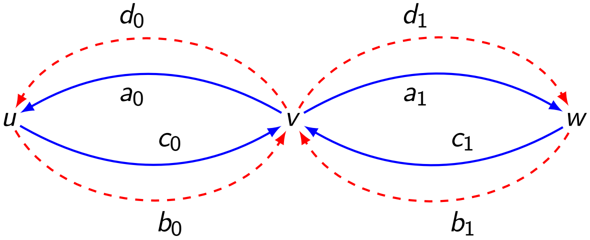


Example of 2-graph



Factorization property: $a_i b_i = d_i c_i$, $\quad a_i b_{1-i} = d_i c_{i-1}$ and $c_i d_i = b_{i-1} a_{i-1}$, $i = 0, 1$.

Example of 2-graph

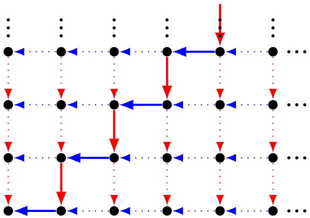


Factorization property: $a_i b_i = d_i c_i$, $a_i b_{1-i} = d_i c_{i-1}$ and $c_i d_i = b_{i-1} a_{i-1}$, $i = 0, 1$.

Vertex Matrices: $A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Diagonal Staircase Representation for 2-Paths

- Because of the **staircase construction** (see the picture for an intuitive idea)
- Infinite paths on a k-graph can also be thought of as
- standard paths in the k-coloured skeleton
- which alternate colours in a **fixed order**
- e.g., blue red blue red blue red . . . .



The C^*-algebra of a k-graph

Let C^*-algebra be the universal C^*-algebra generated by isometries t_λ, $\lambda \in \Lambda$, such that

- $\{t_v | v \ vertex\}$ is a family of orthogonal projections
- Is $s(\lambda) = r(\mu)$, then $t_\lambda t_\nu = t_{\lambda\mu}$
- For $v \in \Lambda_0$, $n \in \mathbb{N}^k$

$$t_v = \sum_{v \in \Lambda^n} t_\lambda t_\lambda^*$$

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Universality means that given a family of operators $\{T_\lambda\}$ in a C^*-algebra A which satisfies the above properties, then there is a unique *-homomorphism

$$\Psi : C^*(\Lambda) \to A$$

such that

$$\Psi(t_\lambda) = T_\lambda.$$

The C^*-algebra of a row-finite k-graph

Moreover

- $C_0(\Lambda^\infty)$ is an abelian sub-algebra of $C^*(\Lambda)$
- The infinite path space characteristic function of the cylinder set associated to the finite path λ corresponds to

$$\{t_\lambda t_\lambda^* | s(\lambda) = s(\mu)\}$$

.

- the span of $\{t_\lambda t_\mu^* | s(\lambda) = s(\mu)\}$ is dense in $C^*(\Lambda)$.

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- We call $\mathcal{O}_n$ the C^*-algebra of the 1-graph $\mathcal{C}_n$
- In the very special case of $\mathcal{C}_1$ the corresponding C^*-algebra $\mathcal{O}_1$ is generated by a single unitary, and is therefore equal to $C(\mathbb{T})$.
- For $n \geq 2$, $\mathcal{O}_n$ is a very interesting C^*-algebra, and has been the subject of a lot of research.

Representations of the C^*-algebra of a k-graph

A representation of $C^*(\Lambda)$ is a *-homomorphism

$$\pi : C^*(\Lambda) \to \mathcal{B}(\mathcal{H}).$$

Not that in particular the C^*-algebra $\pi(C^*(\Lambda))$ contains operators $\pi(t_\lambda) = b_\lambda$ such that

- $\{b_v | v \ vertex\}$ is a family of orthogonal projections (some of them might be zero)
- Is $s(\lambda) = r(\mu)$, then $b_\lambda b_\nu = b_{\lambda\mu}$
- For $v \in \Lambda_0$, $n \in \mathbb{N}^k$

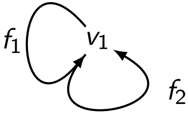
$$b_v = \sum_{v \in \Lambda^n} b_\lambda b_\lambda^*$$

- In the very special case of $\mathcal{C}_1$ the representations of the C^*-algebra $\mathcal{O}_1 \cong C(\mathbb{T})$ have been known for a long time due to the theory of representations of abelian C^*-algebras.
- For $n \geq 2$, $\mathcal{O}_n$ is not abelian, and its representation theory has been the focus of many papers.

Representations of the Cuntz algebra $\mathcal{O}_2$

A representation of $\mathcal{O}_2$ is given by a family of 2 isometries (corresponding to the 2 edges f_1, f_2)

$$\{S_i\}_{i=1,2}$$



$\mathcal{O}_2$

satisfying

$$S_i^* S_j = \delta_{i,j} Id, \ \sum_i S_i S_i^* = Id.$$

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We (F-Gillaspy-Jorgensen-Kang-Packer) generalised some of their results to representations of finite k-graph C^*-algebras. Some of these results have been extended to the row-finite case by (F-Gillaspy-Gonçalves, work in progress).

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The combinatorial nature of k-graph C^*-algebras and their representations (in where the relevant isometries correspond to the finite paths in the k-graph), is fundamental to the theory.

The Natural Faithful Representation

Theorem 1

(FGJKP-2017) Faithful Representation.

- *We give a systematic approach to a constructive representation theory for a Cuntz-Krieger system of generators and relations for k-graphs, known as Λ-semibranching systems, by using only edges (without referring to paths of higher degree).*
- *We give, for a given k-graph Λ, a natural faithful (i.e., zero-kernel) and separable representation of it. This is combinatorial in nature.*

The Projection–Valued Measure Associated to a Representation

Definition 2

Given a representation $\pi = \{t_\lambda\}_{\lambda \in \Lambda}$ of a k-graph C^*-algebra $C^*(\Lambda)$ on a Hilbert space $\mathcal{H}$, we define a projection valued measure P on Λ^∞ by

$$P(Z(\lambda)) = t_\lambda t_\lambda^* \quad \text{for all } \lambda \in \Lambda.$$

Monic Representations: the Projection-Valued Measure

Definition 3

- A representation $\pi : C^*(\Lambda) \to \mathcal{B}(\mathcal{H})$ is monic if

$$\overline{Span\{t_\lambda t_\lambda^* \xi | \lambda \in \Lambda\}} = \mathcal{H}$$

  for some (so called cyclic) vector ξ
- In our work we present examples of monic and non monic representations
- Given a monic representation π of $C^*(\Lambda)$, the projection valued function P defined above gives rise to a measure on Λ^∞ by

$$< P(Z(\lambda))\xi, \xi >=< t_\lambda t_\lambda^* \xi, \xi > \quad \text{for all } \lambda \in \Lambda.$$

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Unitary Equivalence of Monic Representations

We will now present some of our results. The statements below are for finite k-graphs; the row-finite case is similar but requires a different attention since the infinite path space in this case is not compact.

Theorem 4 (FGJKP-2017 for finite k-graphs; rwo-finite case: FGG 2019 work in progress)

Every monic representation π of $C^*(\Lambda)$ is unitarily equivalent to a representation on $L^2(\Lambda^\infty, \mu)$, for some quasi-invariant measure μ on Λ^∞, with $\mu(\Lambda^\infty) < \infty$. Conversely, every representation on $L^2(\Lambda^\infty, \mu)$, with $\mu(\Lambda^\infty) < \infty$, and given in terms of the standard prefixing operators, is monic.

Monic Representations

Theorem 5 (FGJKP-2017 for finite k-graphs; rwo-finite case: FGG 2019 work in progress)

- *So monic representations correspond to quasi-invariant meaures on the infinite path space* Λ^∞
- *Monic representation are unitarily equivalent or disjoint whenever the corresponding measures on the infinite path space are equivalent or disjont.*
- *In other words, the behaviour of the measure reflects the behaviour of the associated representation.*

Universal Monic Representation

Theorem 6 (FGJKP-2017 for finite k-graphs; row-finite case: FGG 2019 work in progress)

Every monic representation π of $C^*(\Lambda)$ is unitarily equivalent to a sub–representation on a universal monic representation of $C^*(\Lambda)$ constructed by using half densities associated to the universal representation of $C(\Lambda^\infty)$.

Atomic Representations

Theorem 7 (FGJKP-2017 for finite k-graphs; row-finite case: FGG 2019 work in progress)

Let $\{S_\lambda\}_{\lambda\in\Lambda}, \{\tilde{S}_\lambda\}_{\lambda\in\Lambda}$ generate purely atomic representations of $C^*(\Lambda)$ on the same Hilbert space $\mathcal{H}$, with associated projection valued measures $P, \tilde{P}$. Then the two representations are unitarily equivalent if and only if the following conditions are satisfied:

(a) $Supp(P) \ = \ Supp(\tilde{P}) \ =: \Omega.$

(b) *For every* $x \in \Omega,\ dim[Range((P(\{x\})))] \ = \ dim[Range((\tilde{P}(\{x\})))]$

Unitary Equivalence of Monic Representations

We will now give an outline of the proof of the above-mentioned theorem below in the case of Cuntz algebras. So we are going to prove

Theorem 8

Every monic representation $\pi : \mathcal{O}_n \to \mathcal{BH}$ is unitarily equivalent to a representation on $L^2(\Lambda^\infty_{\mathcal{O}_n}, \mu_\pi)$. Conversely, every representation on $L^2(\Lambda^\infty_{\mathcal{O}_n}, \mu)$, $\mu(\Lambda^\infty_{\mathcal{O}_n}) < \infty$, is monic.

Unitary Equivalence of Monic Representations

Proof.

Part 1 of the proof, want to show: Every representation on $L^2(\Lambda^\infty, \mu)$, $\mu(\Lambda^\infty_{\mathcal{O}_n}) < \infty$, is monic.
The main idea is to show that the monic vector is the identity function on the infinite path space, thus proving monicity. □

Unitary Equivalence of Monic Representations

Proof.

Part 2 of the proof, want to show: Every monic representation π of $\mathcal{O}_n$ is unitarily equivalent to a representation on $L^2(\Lambda^\infty_{\mathcal{O}_n}, \mu_\pi)$.
This is the more difficult part of the proof. Indeed we will construct an isometry W from $\mathcal{H}$ to $L^2(\Lambda^\infty_{\mathcal{O}_n}, \mu_\pi)$, where μ_π is the measure associated to the previously-introduced projection-valued measure

$$W : L^2(\Lambda^\infty_{\mathcal{O}_n}, \mu_\pi) \to \mathcal{H}, \; W(f) = \pi(f).$$

Then one 'transports' the representation π over to $L^2(\Lambda^\infty_{\mathcal{O}_n}, \mu_\pi)$ via conjugation by W

$$\tilde{S}_i := W^* S_i W,$$

where S_i is the image via π of the i-th generator of $\mathcal{O}_n$. □

Semibranching Function Systems (sbfs)

- Sbfs are a book-keeping devise that was constructed for producing representations of 1-graph algebras; they keep track of the operators associated to edges
- Together with standard prefixing and coding/chopping maps, they give rise to a representation of $C^*(\Lambda)$
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- a systems of partially defined shift operators
- for which the associated Radon-Nikodym derivatives are a. e. $\neq 0$ on their domains

Λ-Semibranching Function Systems

Definition 9

(FGPK-2015) A Λ- Semibranching function system associated to a k-graph algebra $C^*(\Lambda)$ is:

- A collection of P-sbfs (for the 1-graphs corresponding to P), where
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- Compatibility conditions are satisfied.

Ledrappier Example: Products of Matrices

In the Ledrappier 2-Graph, if e.g. $m \leq n$

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

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$$A_1 A_2 = A_2 A_1 = A_1^2 = A_2^2 = D_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$P = A_1^m A_2^n = (A_1 A_2)^m A_2^{n-m} = (A_1 A_2)^m A_2^{2w} A_2^{1 \text{ or } 0} = 4^{m+w-1} D_4 A_2^{0 \text{ or } 1}.$$

Λ-Semibranching System $\Rightarrow$ Representation of $C^*(\Lambda)$

Theorem 10

(FGKP) Suppose we have

- *A* **Λ-semibranching function system** *(Λ-SBFS) on* $L^2(X, \mu)$

Λ-Semibranching System ⇒ Representation of $C^*(\Lambda)$

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- *Then $\{S_\lambda\}_{\lambda \in \Lambda}$ is a* **representation of** $C^*(\Lambda)$

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Corollary 11

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- *form a* Λ-**semibranching function system** *Λ-SBFS*
- *the representation of* $C^*(\Lambda)$ *is* **faithful** *iff the graph is* **aperiodic**