Adaptive Multilevel Monte Carlo Simulation of Stochastic Ordinary Differential Equations

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Problem formulation

For the Ito SDE

$$dX_t = a(X_t, t)\, dt + \sum_{k=1}^{K} b^k(X_t, t)\, dW_t^k, \qquad 0 < t < T, \tag{1}$$

$$X_0 = x_0, \tag{2}$$

and $g : \mathbb{R}^d \rightarrow \mathbb{R}$, approximate $E[g(X_T)]$ to a given accuracy TOL.

W_t is a K-dimensional Wiener process.

Euler Maruyama Method

1. Forward Euler scheme on a grid $t_0 = 0 < t_1 < \ldots < t_N = T$

$$\overline{X}_{n+1} = \overline{X}_n + a(\overline{X}_n, t_n)\Delta t_n + \sum_{k=1}^{K} b^k(\overline{X}_n, t_n)\Delta W_n^k \qquad (3)$$

   gives approximate realisations $\overline{X}_T(\omega)$.

   Here $\Delta t_n = t_{n+1} - t_n$ and $\Delta W_n^k = W_{n+1}^k - W_n^k$

2. Monte Carlo estimate

$$E[g(X_T)] \approx \sum_{i=1}^{M} \frac{g(\overline{X}_T(\omega_i; \Delta t))}{M} \qquad (4)$$

The error contributions

Total error:

$$
\begin{aligned}
&\left| E[g(X_T)] - \sum_{i=1}^{M} \frac{g(\overline{X}_T(\omega_i;\Delta t))}{M} \right| \\
&\leq \left| E[g(X_T) - g(\overline{X}_T)] \right| + \left| E[g(\overline{X}_T)] - \sum_{i=1}^{M} \frac{g(\overline{X}_T(\omega_i;\Delta t))}{M} \right| \\
&\leq TOL_T + TOL_S = TOL
\end{aligned}
$$

Requirement for the time discretization error:

$$|E[g(X_T) - g(\overline{X}_T)]| \leq TOL_T$$

Requirement for the statistical error:

$$\left| E[g(\overline{X}_T)] - \sum_{i=1}^{M} \frac{g(\overline{X}_T(\omega_i;\Delta t))}{M} \right| \leq TOL_S$$

Error Control and Complexity

Weak convergence for smooth drift and diffusion:

$$|E[g(X_T) - g(\overline{X}_T(\cdot;\Delta t))]| = O(\Delta t).$$

$\Delta t \propto TOL$ needed for $|E[g(X_T) - g(\overline{X}_T)]| \leq O(TOL_T)$.

By the Central Limit Theorem, as $M \to \infty$,

$$\sqrt{M}\left(\sum_{i=1}^{M} \frac{g(\overline{X}_T(\omega_i;\Delta t)) - E[g(\overline{X}_T)]}{M}\right) \xrightarrow{D} N\left(0, \sqrt{Var[g(\overline{X}_T)]}\right).$$

$M \propto \frac{1}{TOL^2}$ needed for sufficient probability that $\left|E[g(\overline{X}_T)] - \sum_{i=1}^{M} \frac{g(\overline{X}_T(\omega_i;\Delta t))}{M}\right| \leq O(TOL_S)$.

Computational complexity $= M\frac{T}{\Delta t} \propto 1/TOL^3$.

Giles' multilevel idea 2006

On a hierarchy of uniform grids $\Delta t_\ell = \Delta t_0/2^\ell$, $\ell = 0, \ldots, L$, let $g_\ell = g(\overline{X}_T(\cdot; \Delta t_\ell))$.

Step 1 Write the telescopic sum

$$E[g_L] = E[g_0] + \sum_{\ell=1}^{L} E[g_\ell - g_{\ell-1}].$$

Step 2 Now use $L+1$ batches, each with M_ℓ independent realizations, $\ell = 0, \ldots, L$ to create the estimator

$$A(M_0) = \sum_{i_0=1}^{M_0} \frac{g_0(\omega_{i_0})}{M_0} + \sum_{\ell=1}^{L} \sum_{i_\ell=1}^{M_\ell} \frac{(g_\ell - g_{\ell-1})(\omega_{i_\ell})}{M_\ell}.$$

Giles' multilevel idea 2006

Strong convergence order 1/2 :
$Var(g_\ell - g_{\ell-1}) = O(\Delta t_\ell) = O(\Delta t_0/2^\ell)$
Optimal choice $M_\ell = M_0/2^\ell$ so that

$$Var(A) = \frac{Var(g_0)}{M_0} + \sum_{\ell=1}^{L} \frac{Var(g_\ell - g_{\ell-1})}{M_\ell} = \frac{O(1)}{M_0}(1 + \Delta t_0 L)$$

To achieve $Var(A) = O(TOL^2)$ take

$$M_0 \propto (TOL^{-2}(1 + \Delta t_0 L))$$

giving the total work to achieve accuracy TOL

$$Work \propto \sum_{\ell=0}^{L} \frac{M_\ell}{\Delta t_\ell} \quad = \quad O\left((1+L)\frac{M_0}{\Delta t_0}\right)$$

$$= O\left((\log_2(\Delta t_0/TOL))^2 \; TOL^{-2} \right)$$

When the time error dominates. . .

Example Stopped diffusion:

$$dX_t = \begin{cases} \frac{11X_t}{36}dt + \frac{X_t}{6}dW_t, & \text{for } t \in [0,2] \text{ and } X_t \in (-\infty, 2) \\ 0 \quad (\text{Stopped!}), & \text{if } X_t = 2, \end{cases}$$

$$X_0 = 1.6,$$

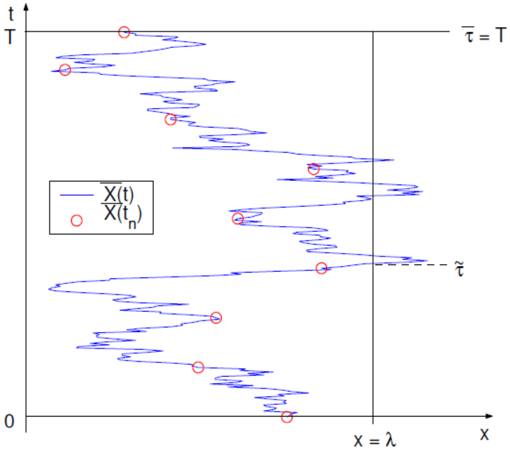
$$g(x,t) = x^3 e^{-t}$$

compute

$E[g(X_\tau, \tau)]$ τ stopping time

- Weak convergence uniform grid: $O(1/\sqrt{N})$.

Hitting error



Adaptivity

Given TOL_T, use adaptive refinements to generate grids $t_0 = 0 < t_1(\omega) < \ldots < t_N = T$ to create realizations $\overline{X}_T(\omega; \Delta t(\omega))$.

How? Adaptive refinements start from a coarse initial grid, and *iteratively*

(1) computes solution and and an estimate, r_n, of each time step's error contribution, by residuals weighted with dual solution (and here exit probabilities)

(2) as long as $\max_n r_n \geq C_S \frac{TOL_T}{E[N]}$,

(3) refine all time steps s.t. $r_n \geq C_R \frac{TOL_T}{E[N]}$, refine sampling by Brownian bridges, and go to (1)

How? Poster session

- Weak convergence **adaptive grid:** $O(1/N)$.

Idea adaptive multilevel algorithm

Multilevel: Grid hierarchy defined by $TOL_{T,\ell} = TOL_{T,L} 2^{L-\ell}$.

For each realisation, ω, on level ℓ:

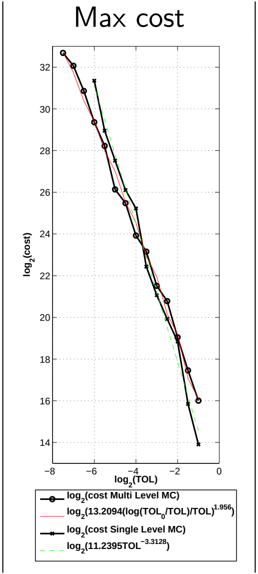
compute by adaptive algorithm $(g_\ell - g_{\ell-1})(\omega)$

by succesively halving the tolerance from $TOL_{T,0}$ to $TOL_{T,\ell-1}$ and $TOL_{T,\ell}$

Again $M_\ell = 2^{-\ell} M_0$

and $A(M_0) = \sum_{i_0=1}^{M_0} \frac{g_0(\omega_{i_0})}{M_0} + \sum_{\ell=1}^{L} \sum_{i_\ell=1}^{M_\ell} \frac{(g_\ell - g_{\ell-1})(\omega_{i_\ell})}{M_\ell}$.

Experimental Complexity



Experimental Accuracy

Error