Bayesian Sensitivity Analysis of a Large Nonlinear Model

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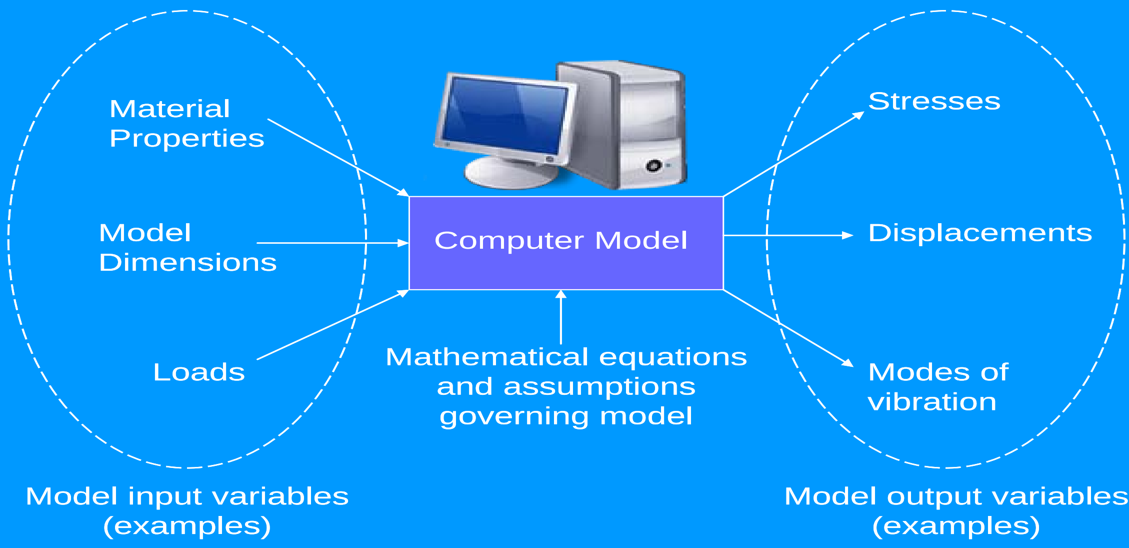
Overview

- Uncertainty and sensitivity analysis
- The Bayesian approach
- Heart valves
- The finite element model
- Sensitivity analysis results
- Conclusions
- Questions and comments

Uncertainty in Modelling



Aleatoric

- Arising from inherent variability
- Machining tolerance, operating conditions
- Cannot be reduced

Epistemic

- “Model Imperfections”
- Simplifications, precise information unavailable
- Can be reduced

Sensitivity Analysis

- “How do individual model inputs contribute to the uncertainty in the output?”
- Why:
  - Increase robustness of model
  - Design optimisation
  - Identify parameters that require further research
  - Model simplification – eliminating variables
  - Greater understanding of model and variable interactions

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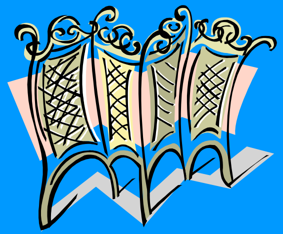
Sensitivity Analysis

Screening
(qualitative ranking)

Local SA
(Linear models,
small perturbations)

Global SA
(Often Monte Carlo)

Least Informative



Most Informative

Increasing computational cost

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Problems…

- Complex simulations can require a significant time for a single run
- Monte-Carlo techniques require many runs
- SA for several input variables can be unfeasible

A Solution – Bayesian Data Modelling

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The Bayesian Approach

- Model treated as unknown function *f*(x)
- Input parameters represented as probability distributions (uniform or Gaussian for tractability)
- Gaussian process regression (GPR) used to build a metamodel from small number of model runs
- Sensitivity analysis data inferred directly from posterior distribution
- Application of GPR allows SA data to be collected for many fewer model runs, at comparable accuracy

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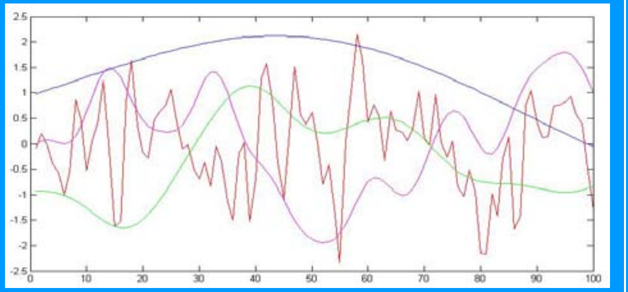
Gaussian Process Regression

Prior assumptions

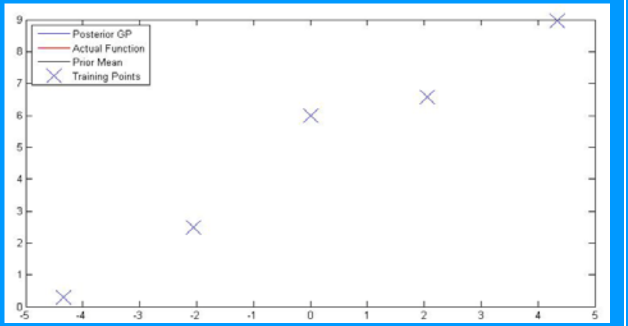
$$E\{f(\mathbf{x}) \mid \beta\} = \mathbf{h}(\mathbf{x})^T \beta$$

$$\mathrm{cov}\{f(\mathbf{x}_i), f(\mathbf{x}_j) \mid \sigma^2, B\} = \sigma^2 \exp\{-(\mathbf{x}_i - \mathbf{x}_j)^T B (\mathbf{x}_i - \mathbf{x}_j)\}$$



Training data

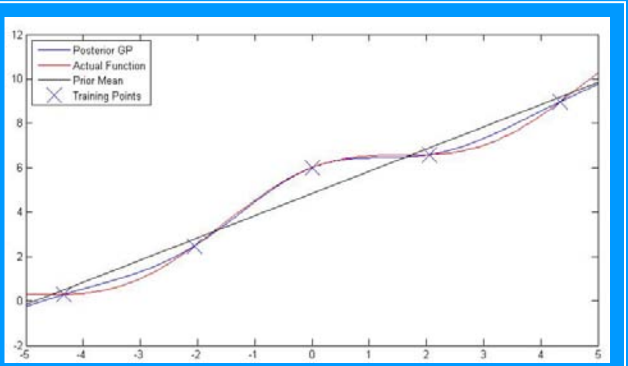
$$\mathbf{X} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$$
$$\mathbf{y} = \{y_1, y_2, \ldots, y_n\}$$



Hyperparameter estimation, condition on training data

Posterior distribution

$$[f(\mathbf{x}) \mid B, \mathbf{y}] \sim t_{n-q}\{m^*(\mathbf{x}), \hat{\sigma}^2 c^*(\mathbf{x}, \mathbf{x}')\}$$



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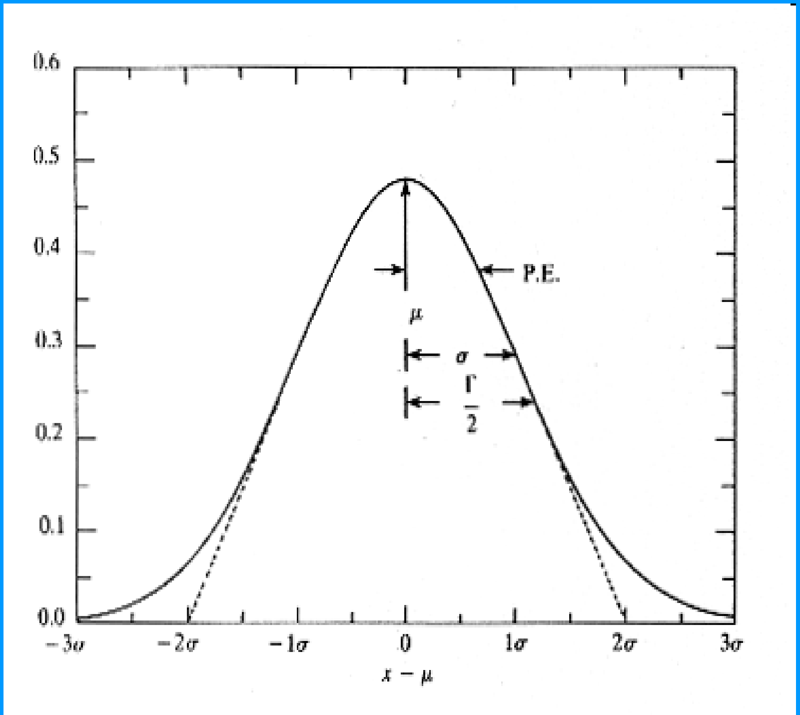


Uncertainty in output

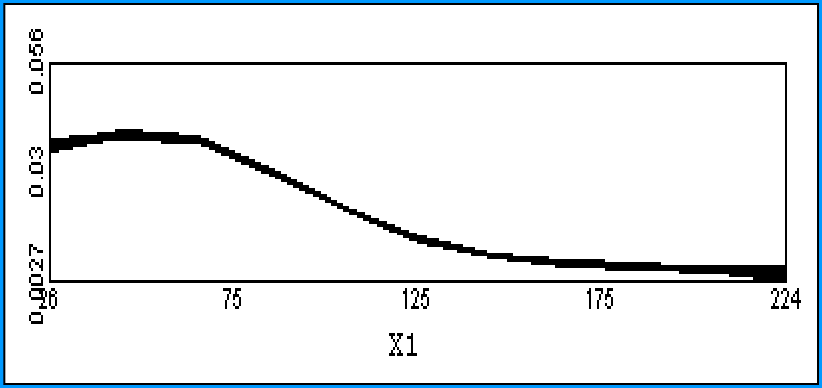
- Mean $E^*\{E(Y)\}$
- Variance

Posterior distribution

$$[f(\mathbf{x}) \mid B, \mathbf{y}] \sim t_{n-q}\{m^*(\mathbf{x}), \hat{\sigma}^2 c^*(\mathbf{x}, \mathbf{x}')\}$$

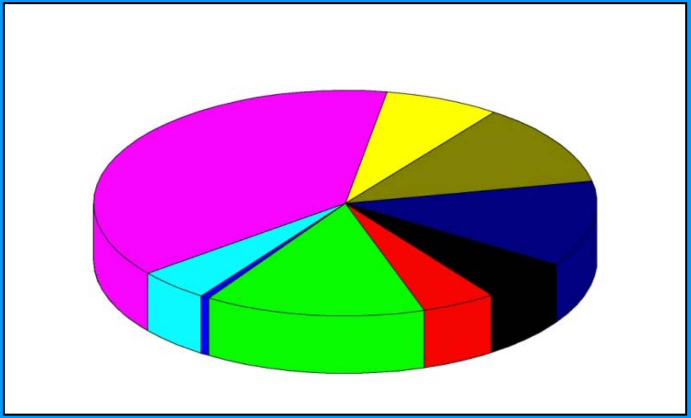
Main effects & interactions

$$E^*\{E(Y \mid \mathbf{x}_p)\} = \int_{\chi_{-p}} m^*(\mathbf{x}) dG_{-p|p}(\mathbf{x}_{-p} \mid \mathbf{x}_p)$$

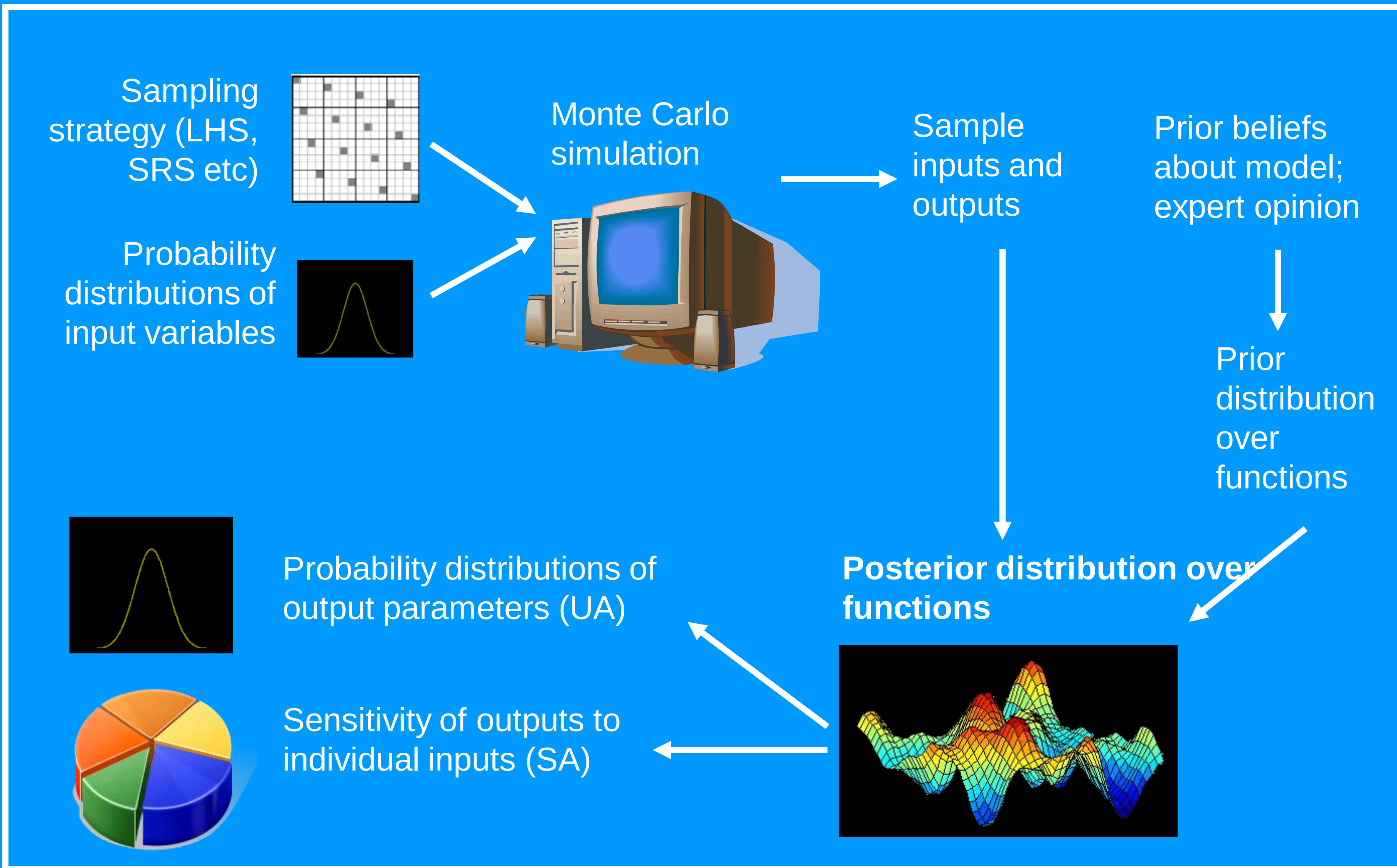


Sensitivity indices

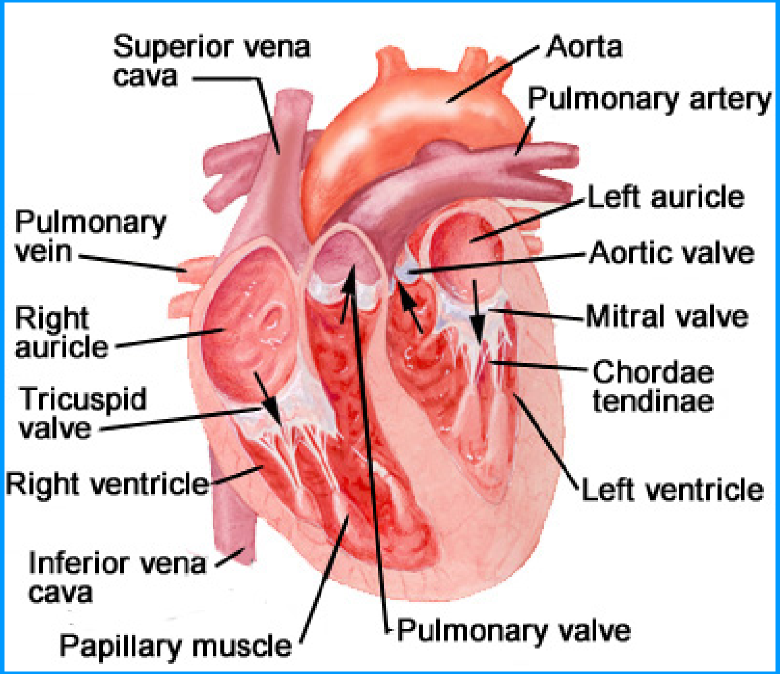
$$E^*[\mathrm{var}\{E(Y \mid X_p)\}]$$
$$= E^*[E\{E(Y \mid X_p)^2\}] - E^*\{E(Y)^2\}$$

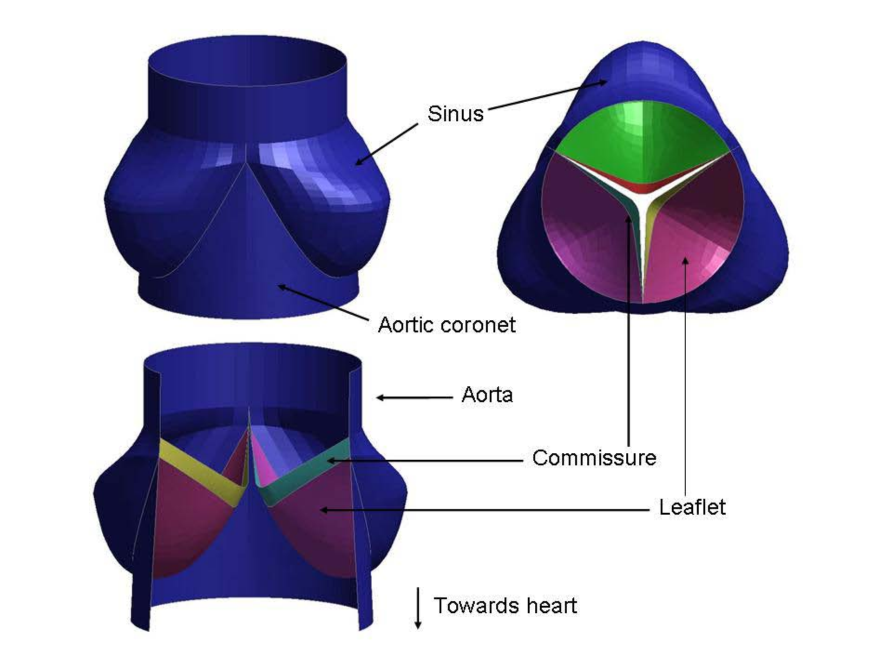


A Bayesian Approach



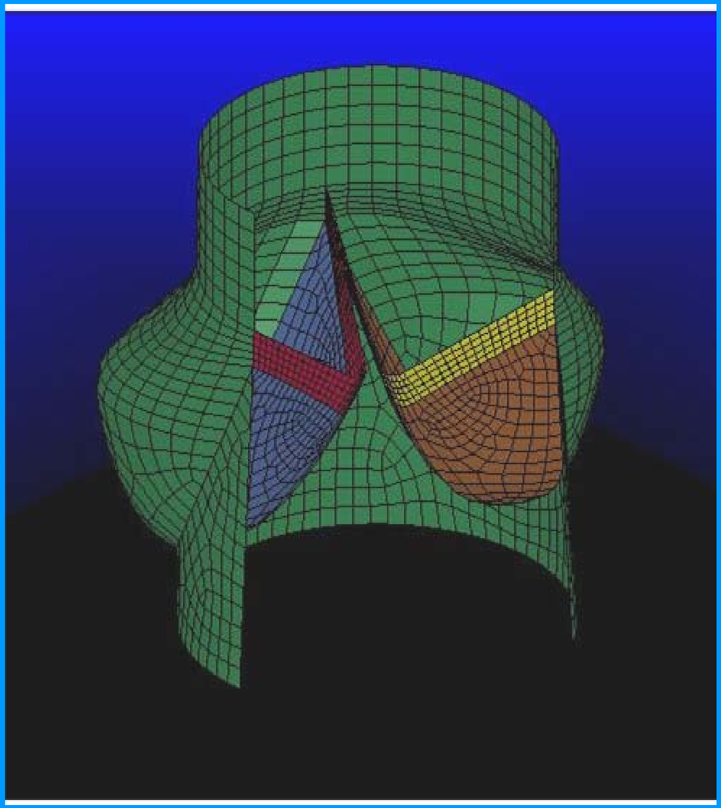
The Aortic Valve





- Opens and closes ~10,000 times per day
- The most common heart valve to fail
- Bioprosthetic design driven by understanding mechanics of natural valve

The Finite Element Model

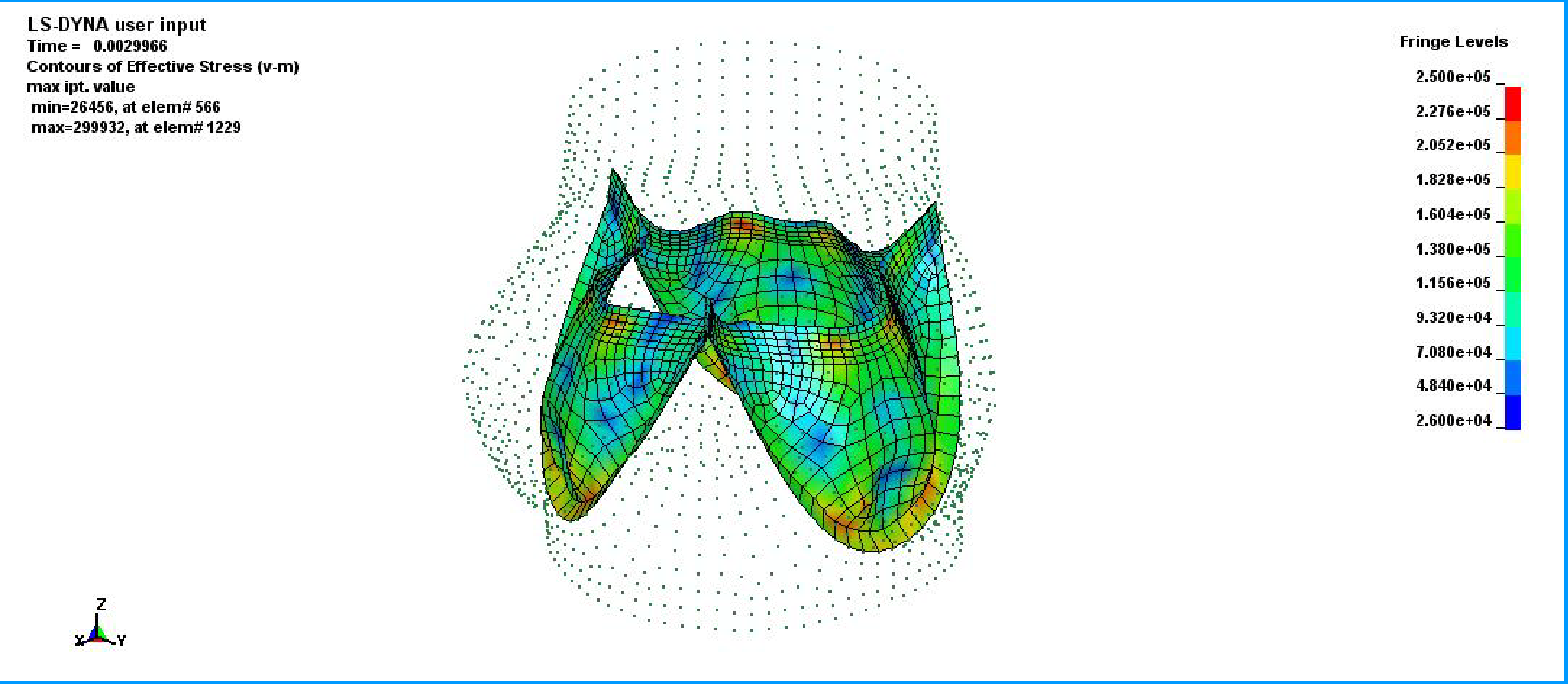


- Dynamic model created in Ansys, solved in LS-Dyna
- 3609 Belytschko-Tsay linear elastic shell elements
- Pressure loading from cardiac cycle
- Simulation covers opening of valve only (0.1s from positive z-pressure)
- Output frequency: 1000 states/sec

Model Limitations

- Linear-elastic isotropic material model
  - Considered sufficient for initial investigation
  - Limited data for hyperelastic SA
- Pressure loading unrealistic
  - ALE fluid loading preferable
- Leaflet thickness constant
  - Varying thickness in reality

The FE model - Outputs



Von Mises stress fringe plot, typical parameter set

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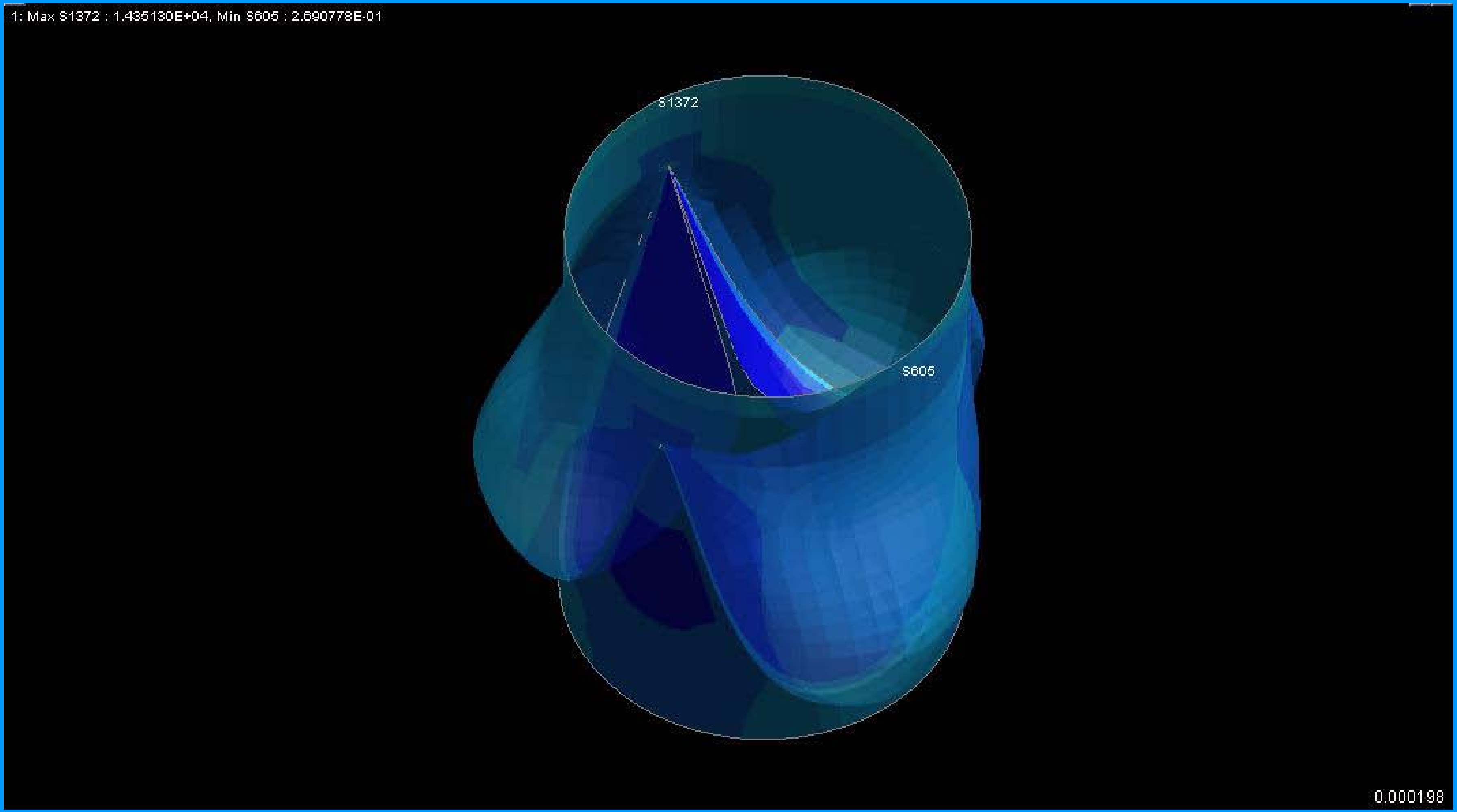

Animated plot of Von Mises stress on valve opening

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The FE model – Outputs

- Stress
  - Max. principal
  - Von Mises
  - Bending
  - Etc…
- Displacements
- Velocities
- Accelerations
- Forces
- Strains
- Buckling
- Etc…

- Maximum stress
- Maximum leaflet displacement
- Opening speed
- Buckling of leaflet on opening

All require dimensional reduction – output must be one-dimensional (at the moment)

Red outputs were investigated in this study

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Uncertain Inputs

- Material (2)
- Shell thickness (3)
- Geometrical (3)

- Inputs assumed Gaussian
- Aleatoric and epistemic uncertanties present

Parameter	Name	Mean	Standard Deviation	Data source
Elastic modulus of sinus (MPa)	*Es*	2.45	0.8	Lit. review
Thickness of sinus shells (mm)	*Ts*	1.3	0.325	Lit. review
Elastic modulus of leaflet (MPa)	*El*	2.05	0.65	Lit. review
Thickness of leaflet shells (mm)	*Tl*	0.35	0.0085	Lit. review
Thickness of commissure shells (mm)	*Tc*	1	0.25	Lit. review/Thubrikar
Radius of base (mm)	*Rb*	12.7	0.675	Thubrikar
Angle of coaptation (degrees)	Φ	31	3	Thubrikar
Leaflet separation in relaxed state (ratio)	*Fltrad*	6	2	Subjective

The Sensitivity Analysis

- Maximin latin hypercube sampling
- 250 model runs, approx 4 hours
- Squared-exponential covariance function (assumes smooth response)
- Inputs assumed uncorrelated
- Gem-SA used for DOE and analysis

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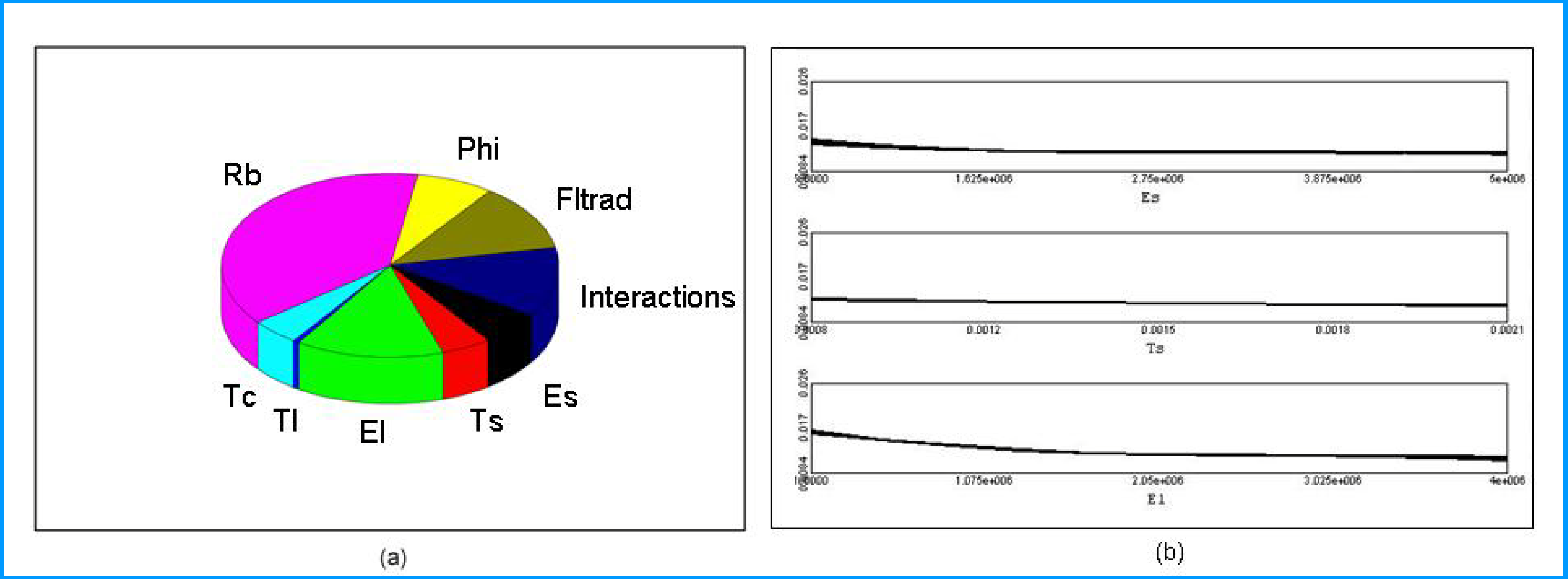
Results - Uncertainties

Parameter	**Posterior mean (μ)**	**Posterior standard deviation (σ)**	**σ/ μ (%)**
Dispmax	0.0125	0.00160562	12.8
Thalfop	0.00194	0.00122988	63.4
Sigmaxmax	635878	354868.99	55.8
Wig	0.161268	0.01573881	9.8

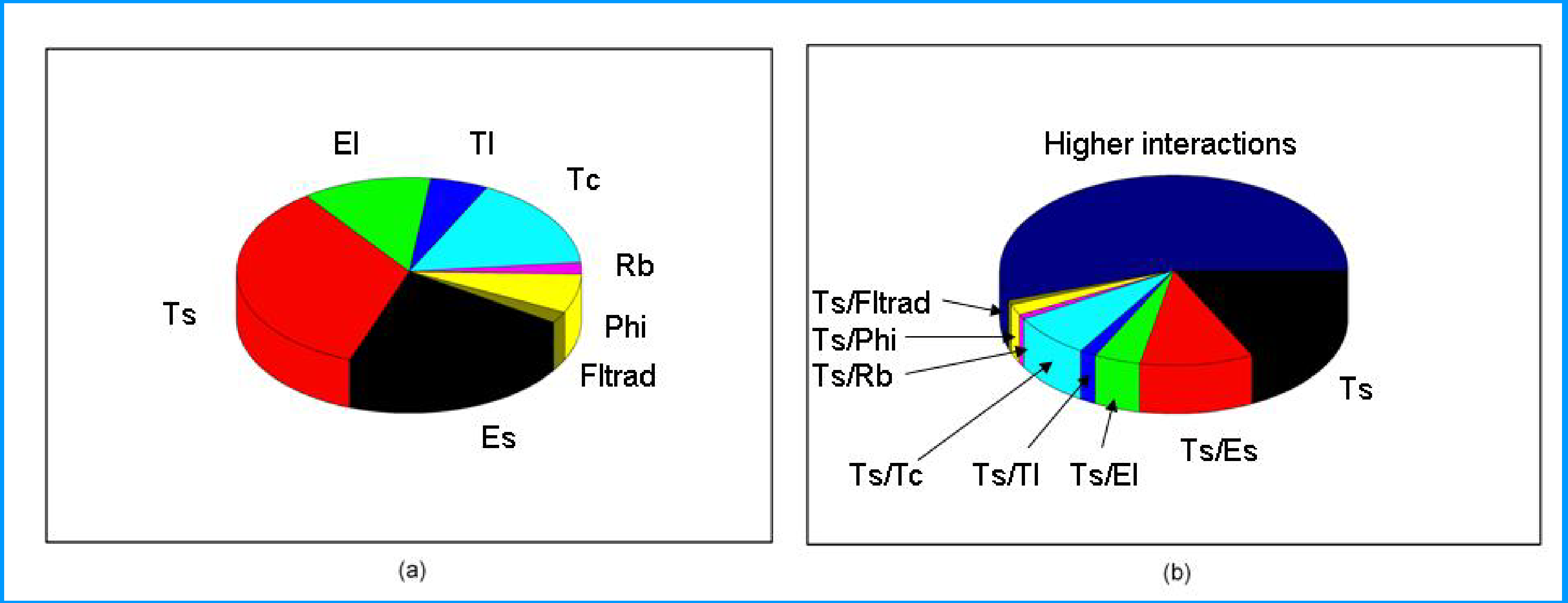
Sensitivity Results



Sensitivity indices and main effect plots for output *dispmax*

- Base radius is dominant input
- Interactions account for only 10% of output variance
- Nonlinearities evident

Sensitivity Results (Opening Speed)



sensitivity indices: (a) Total effect indices; (b) Interactions involving *Ts*

- Approx. 70% variance due to interactions
- Flexibility of sinus important in opening valve
- Higher interactions evidence of model complexity

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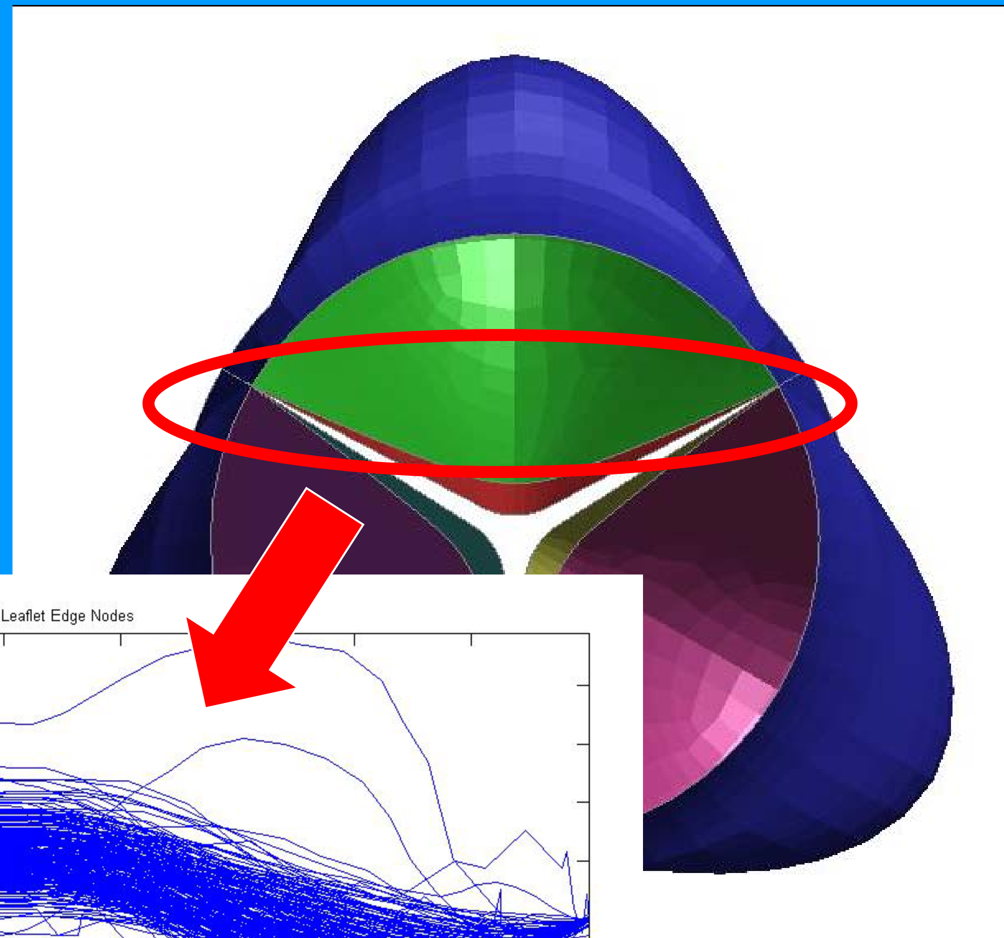
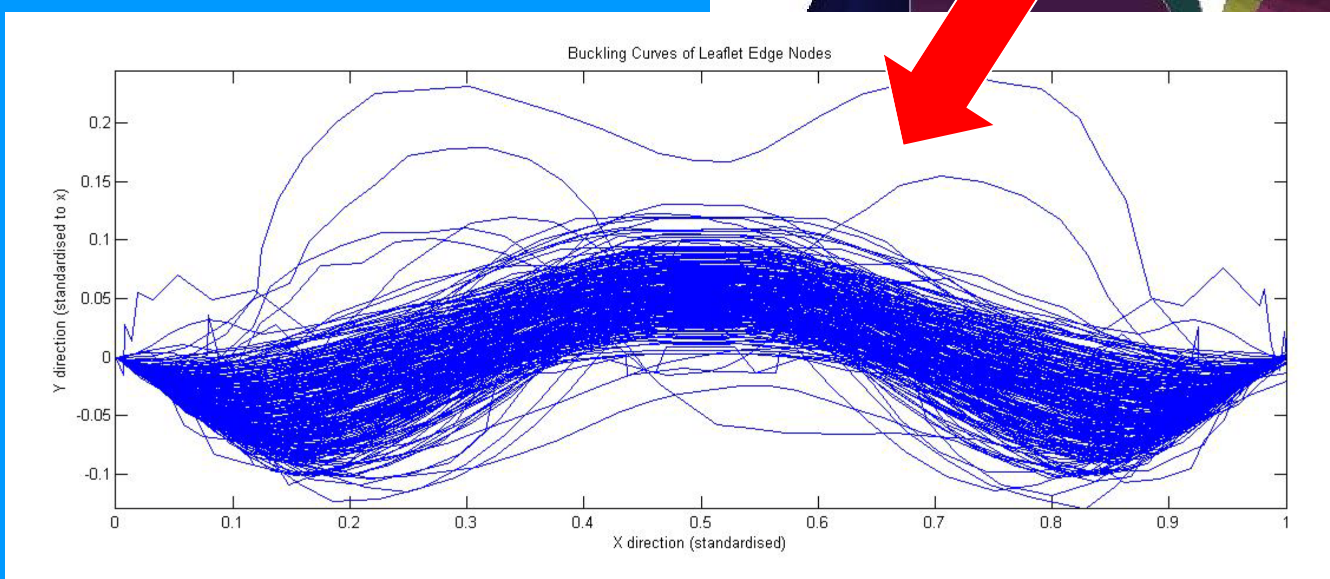
Buckling of leaflet

- Evidence of instabilities in model
- Generally smooth opening





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Conclusions

- Bayesian sensitivity analysis allows detailed insight into large, nonlinear uncertain models.
- Analysis of heart valve shows that output uncertainty can be substantial in FE models.
- Assumptions used (smoothness of model, input distributions etc), thus uncertainty results uncertain! However, good indicator.
- Non-linearities inherent – global UA necessary.
- 7 inputs varied, out of thousands, thus uncertainty undoubtedly greater than shown here
- FE models should be subjected to uncertainty analysis where appropriate (for example, biomaterials).

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Developments

- Bifurcations – Gaussian Process Regression Trees
- Fluid-Structure Interaction
- Hyperelastic etc. material properties