# Mathematics and Computation of Sediment Transport in Open Channels

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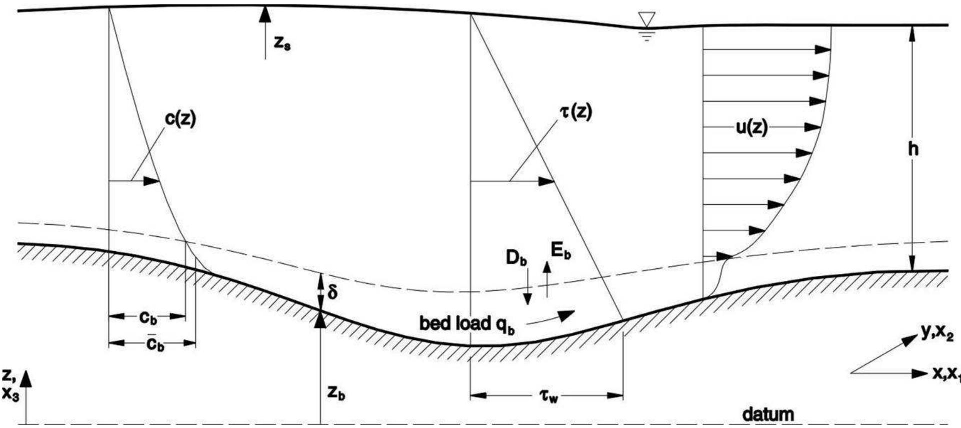
One major problem to people living downstream of Mount St. Helens is the high sedimentation rates resulting from stream erosion of the volcanic deposits. Streams are continuously downcutting channels, eroding their banks, and eating away at the avalanche and lahar deposits. This material is eventually transported downstream and deposited on the streambeds, decreasing the carrying capacity of the channels and increasing the chances of floods. – USGS Photo by Lyn Topinka, February 22, 1982

# Mathematical Models for Sediment Transport

The computer/mathematical simulation of open channels includes the following major components:

- Flow calculation (coupled surface and subsurface fluid flow)
  - Saint-Vernant equations
  - Darcy flow
  - Interface conditions that link the two systems
- Sediment calculation
  - Sediment transport (math and computation)
  - Bed change
  - Bed material sorting

# Components in Math Simulation



Conservative form of the Saint-Venant equations:

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F} = \mathbf{S},$$

where

$$\mathbf{U} = \begin{bmatrix} h \\ hu \\ hv \end{bmatrix}; \quad \mathbf{F} = [\mathbf{E}, \mathbf{G}];$$

$$\mathbf{E} = \begin{bmatrix} hu \\ hu^2 + gh^2/2 \\ huv \end{bmatrix}; \quad \mathbf{G} = \begin{bmatrix} hv \\ huv \\ hv^2 + gh^2/2 \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} 0 \\ gh(S_{0x} - S_{fx}) \\ gh(S_{0y} - S_{fy}) \end{bmatrix} + (R_f - I_n) \begin{bmatrix} 1 \\ u \\ v \end{bmatrix}.$$

Force due to the change of bottom slop:

$$(S_{0x}, S_{0y}) = -\left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right),$$

and friction (Manning's formula)

$$S_{fx} = \frac{n^2 u \sqrt{u^2 + v^2}}{K_n^2 h^{4/3}}; \quad S_{fy} = \frac{n^2 v \sqrt{u^2 + v^2}}{K_n^2 h^{4/3}}$$

where $n$ is the Manning's roughness coefficient.

# Subsurface Flow

- The Darcy's law for one-dimensional, unsteady, vertical flow is given by

$$q = -K(\theta)\left(\frac{\partial h(\theta)}{\partial z} + 1\right).$$

  The positive z-direction points upward.

- The mass conservation:

$$\frac{\partial \theta}{\partial t} + \nabla \cdot q = 0,$$

  where $\theta$ is the volumetric water content.

- A 3D porous media flow model should be employed for a more accurate simulation of the subsurface flow.

# Boundary and Initial Conditions: Surface Flow

- Boundary conditions on lateral sides are
  - wall condition (fix boundary)
  - free flow (moving boundary)
  - given data (such as flow rate) at inflow boundary
- Initial condition for flux. Water depth and velocity are expected to satisfy the Manning's formula:

$$S_{0x} = \frac{n^2 u \sqrt{u^2 + v^2}}{K_n^2 h^{4/3}}; \quad S_{0y} = \frac{n^2 v \sqrt{u^2 + v^2}}{K_n^2 h^{4/3}}$$

# Surface and Subsurface Coupling

- Continuity of velocity in normal component:

$$\mathbf{u} \cdot \mathbf{n}_1 + \mathbf{q} \cdot \mathbf{n}_2 = 0,$$

- $I_n$ is the infiltration rate which appears in the surface flow, as well as in the subsurface flow as a boundary condition at the bed $z = z_b(x, y, t)$.
- Force continuity:

$$-\rho n_1^t \sigma(u, p) \cdot n_1 = s,$$

  where $s$ is the pressure of the subsurface flow on the interface.
- Water depth is $h(t, x, y) = z_t - z_b$ so that

$$s = -\rho g h(t, x, y).$$

# The Green-Ampt Formula

The model comes from mass conservation and the Darcy's law:

$$q(t,z) = -K\left(\frac{\partial h(t,z)}{\partial z} + 1\right), \quad z_b - L(t) < z < z_b,$$

$$\frac{\partial \theta}{\partial t} + \frac{\partial q(t,z)}{\partial z} = 0, \qquad z_b - L(t) < z < z_b.$$

- Dirichlet boundary at two ends:

$$h(t, z_b) = h_0(t) \quad \text{(surface water depth)}$$

$$h(t, z_b - L(t)) = -\psi \quad \text{(capilarity)}$$

- Neumann at the wet (moving) front

$$\frac{\partial L(t)}{\partial t} \Delta\theta = -q(t, z_w).$$

# The Green-Ampt Formula with Uncertainty

Solving the equation gives an ODE for the moving wet front:

$$L'(t) = \left((\psi + h_0(t))L^{-1}(t) + 1\right) K/\Delta\theta.$$

Big uncertainty on

- Manning's coefficient $n$
- Hydraulic conductivity $K$
- Suction head $\psi$

# Numerics for Surface Flow

- Integrate the governing equation over the control volume $K$ gives

$$\int_K \frac{\partial \mathbf{U}}{\partial t} dx + \int_K \nabla \cdot \mathbf{F} dx = \int_K \mathbf{S} dx.$$

- Apply the divergence theorem one obtains

$$\int_K \frac{\partial \mathbf{U}}{\partial t} dx + \int_{\partial K} \mathbf{F} \cdot \mathbf{n} ds = \int_K \mathbf{S} dx.$$

- With $\mathbf{U}_K = \frac{1}{|K|} \int_K U dx$, one arrives at

$$\frac{\Delta \mathbf{U}_K}{\Delta t} |K| = - \int_{\partial K} \mathbf{F} \cdot \mathbf{n} ds + \int_K \mathbf{S} dx.$$

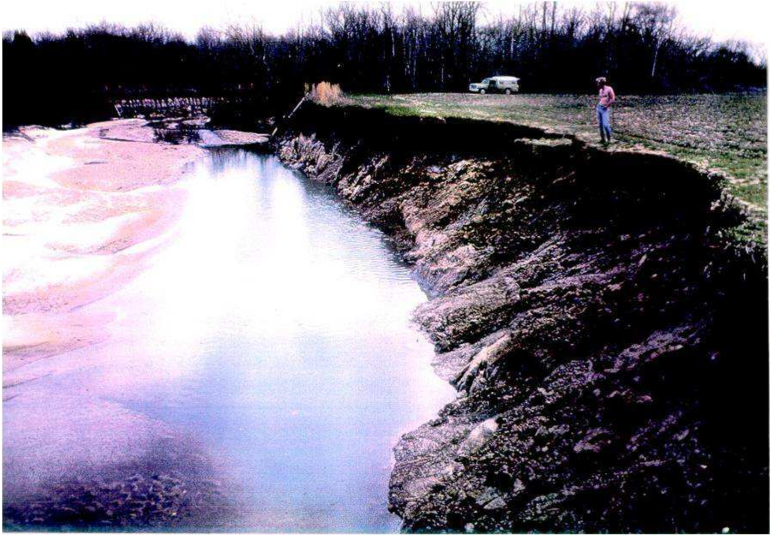
# Numerics for Surface Flow

- The boundary integrals involve the normal component of the flux variable which has to be determined.
- Suppose that there is already a way to determine these boundary fluxes. Then the numerical scheme can be written as

$$\Delta \mathbf{U}_K = -\frac{\Delta t}{|K|} \sum_{i=1}^{3} \mathbf{F}_i^* \cdot \mathbf{n}_i |\ell_i| + \frac{\Delta t}{|K|} \int_K \mathbf{S} dx.$$

- The boundary flux can be approximated by approximating the corresponding Riemann problem.

# Modeling of Sediment Transport



# Classification of Grain Classes

- The range of transport material, say between 0.002*mm* and 1048*mm*, is divided into $N$ grain classes or bins that contain adjacent, nonoverlaping portions of the grain size spectrum.
- The default grain classes are based on a standard log base 2 scale where the upper bound of each class is twice the upper bound of the adjacent, but smaller class.
- All the particles in each grain class are presented by a single and representative grain size. We use the geometric mean of the grain class (the square root of the product of the upper and lower bounds) to represent the grain size for each bin.

# Classification of Grain Classes

For example, for the transport material ranging between $smm$ and $Smm$, the grain class $j$ contains particles of size between $s2^{j-1}mm$ to $s2^{j}mm$. The geometric mean of class $j$ grains would be

$$d_j = s2^{j-1/2}.$$

With $s = \sqrt{2}10^{-3}$, we see that the grain size $d_1 = 0.002mm$ and grain size $d_{20} = 1048mm$. In other words, the sediment material is divided into 20 grain classes for the ranges under consideration.

# Sediment Transport in Suspended Zone

For each grain class $j$, let $C_j$ be the average concentration for class size $j$ in the suspended region.

$$\frac{\partial(hC_j)}{\partial t} + \frac{\partial(hUC_j)}{\partial x} + \frac{\partial(hVC_j)}{\partial y} = \Phi_{s,j},$$

where the source/sink term $\Phi_{s,j}$ is given by

$$\Phi_{s,j} = \frac{\omega}{\beta}\left(\frac{\beta p^*_{b,j} q_{b,j}}{6.11 U_* k_{sj}} - C_j\right),$$

- $q_{b,j}$: bed-load transport capacity
- $p^*_{b,j}$: portion of the size class $j$ in the mixing layer

- $\beta$: a parameter given by:

$$\beta = \frac{1}{h-a}\left(\frac{a}{h-a}\right)^{z_*}\int_a^h\left(\frac{h-z}{z}\right)^{z_*}dz$$

with

$$z_* = \frac{\omega}{\kappa\sqrt{U^2+V^2}}, \quad \kappa = 0.4,$$

- $U_*$: shear velocity which will be discussed in coming Sections.

# Sediment Transport for Bed Load

- Bed load transport equation:

$$\frac{\partial(\theta_j(1-\varepsilon_j))}{\partial t} + \nabla \cdot (p_j q_{b,j}) = 0,$$

  where $\varepsilon_j$ is the porosity of bed load for size class $j$.
- $\theta_j$: thickness of the mixing layer for size class $j$ as a result of the bed-load (rolling of sediments along the river bed).

# Sediment Transport for Total Bed Material Load

The total bed material load simulation contains two components:

- the accumulation/dispersion of sediment from/to the suspended region
- the transport of sediments along the river bed.

The first component comes from the suspended load:

$$(1-\varepsilon)\frac{\partial z_{b,j}}{\partial t} = -\Phi_{s,j},$$

where $\varepsilon$ is the porosity of the bed material.
The second component comes from the bed load. The combined version gives

$$\frac{\partial (z_j(1-\varepsilon_j))}{\partial t} + \nabla \cdot (p_j q_{t,j}) = -\Phi_{s,j}.$$

# Bed Material Sorting

- Update the bed elevation at time level $t_{n+1}$ as follows:

$$z_b(x, y, t_{n+1}) = z_b(x, y, t_n) + \sum_{j=1}^{N} \tilde{\delta}_j(x, y, t_{n+1}).$$

- How can we update the material gradation function $p_j$?
- Amount of size class $j$ sediment at time level $t_{n+1}$ is given by

$$A_j = p_j(t_n)(z_b(t_n) - z_{ne}) + \tilde{\delta}_j(t_{n+1}),$$

and the total amount of sediment at time level $t_{n+1}$ is

$$A = z_b(x, y, t_n) - z_{ne}(x, y) + \sum_{j=1}^{N} \tilde{\delta}_j(x, y, t_{n+1}).$$

- Thus, the material gradation at time level $t_{n+1}$ is

$$p_j(x, y, t_{n+1}) = \frac{A_j}{A} = \frac{p_j(t_n)(z_b(t_n) - z_{ne}) + \tilde{\delta}_j(t_{n+1})}{z_b(t_n) - z_{ne} + \sum_{j=1}^{N} \tilde{\delta}_j(t_{n+1})}.$$

# Bed-load Formulas

Non-dimensional form:

$$\Phi_B = \frac{q_B}{d\sqrt{(s-1)gd}}.$$

- Kalinske-Frijlink formula

$$q_B = 2d_{50}\sqrt{\frac{\tau_b}{\rho}}\exp\left(\frac{-0.27(s-1)d_{50}\rho g}{\tau_b'}\right),$$

- Meyer-Peter formula:

$$\Phi_B = 8(\theta' - \theta_c)^{3/2},$$

where

- $\theta'$: effective Shields parameter given by $\theta' = \frac{\tau_b'}{\rho(s-1)gd}$
- $\tau_b'$: effective shear stress
- $\theta_c$: critical Shields parameter

# Bed-load Formulas

- Einstein-Brown formula:

$$\Phi_B = 40K(\theta')^3,$$

where

$$K = \sqrt{\frac{2}{3} + \frac{36\nu^2}{(s-1)gd_{50}^3}} - \frac{6\nu}{d_{50}^{1.5}\sqrt{(s-1)g}}.$$

# Shields diagram

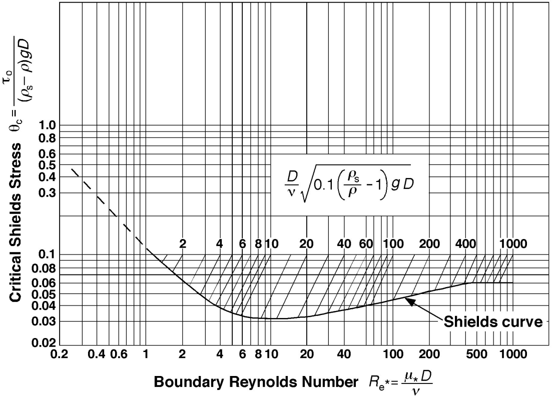



Figure: Shields diagram that defines the critical value $\theta_c$ as a function of the Reynolds number.