Uncertainty Quantification for Hybrid Dynamical Systems

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with José Miguel Pasini
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Hybrid dynamical systems are a useful and common abstraction of systems with multiple modes of operation. Each mode has a different set of governing differential equations. Challenging for analysis: give rise to new phenomena like Zeno behavior. Examples include bouncing ball, water tank etc.
Problem Definition

The system switches from one mode to another if guard condition is satisfied,

\[ \dot{x} = f(x, \lambda) : \text{if } x \in \text{Domain}(q) \]

\[ \dot{x} = g(x, \lambda) : \text{if } x \in \text{Domain}(q') \]

Reset condition: \(x^+ = h(x^-)\) (e.g. bouncing ball).

For simplicity consider (without loss of generality),

\[ \dot{x} = f(x, \lambda) : \text{if } x \geq 0 \]

\[ \dot{x} = g(x, \lambda) : \text{if } x < 0. \]

\(\lambda\) is a vector of random parameters - Fast Uncertainty propagation.
Comparison of various methods

10000 Monte Carlo samples is considered as baseline. Compared different methods

- Monte Carlo (MC)
- Quasi Monte Carlo (QMC)
- Generalized Polynomial Chaos (gPC) for hybrid systems
- Probabilistic Collocation Method (PCM) - 9th order approximation
- Multi Element Galerkin Polynomial Chaos (ME-gPC)
Proposed gPC method: New Methodology

Method 1:

\[
\dot{x} = f(x, \lambda) : \text{if } x \geq 0 \\
\dot{x} = g(x, \lambda) : \text{if } x < 0.
\] (2)

Define \(\theta(x)\):

\[
\theta(x) = 1 : \text{if } x \geq 0 \\
\theta(x) = 0 : \text{if } x < 0.
\] (3)

Then,

\[
\dot{x} = \theta(x)f(x, \lambda) + (1 - \theta(x))g(x, \lambda)
\] (4)

Expand \(x(t, \lambda) = a_i(t)H_i(\lambda)\) (Einstein notation).
Proposed gPC method: New Methodology

\[ \dot{a}_i H_i = \theta(a_i H_i) f(a_i H_i, \lambda) + (1 - \theta(a_i H_i)) g(a_i H_i, \lambda) \]  
(5)

Multiply above relation by \( \rho(\lambda)H_k(\lambda) \) and integrate,

\[ \dot{a}_k = \int_{R_1(t)} f(a_i H_i, \lambda) \rho(\lambda) H_k(\lambda) d\lambda + \int_{R_2(t)} g(a_i H_i, \lambda) \rho(\lambda) H_k(\lambda) d\lambda \]  
(6)

\( R_1(t) = \{ \lambda : a_i H_i \geq 0 \} \) and \( R_2(t) = R - R_1(t) \)

\( R_1(t) \) can be computed by looking at the zeros of \( a_i(t) H_i(\lambda) \)
Computing $R_1(t)$ and $R_2(t)$

Zeros of the polynomial give boundaries of region (1D)
Alternative gPC approach

Method 2:

\[
\dot{x} = \frac{1}{2} \left[ 1 + \tanh\left( \frac{x}{\epsilon} \right) \right] f(x, \lambda) + \frac{1}{2} \left[ 1 - \tanh\left( \frac{x}{\epsilon} \right) \right] g(x, \lambda). \tag{7}
\]

Expand \( x(t, \lambda) = a_i(t)H_i(\lambda) \) to give,

\[
\dot{a}_k = \int_R \left[ \frac{1}{2} \left[ 1 + \tanh\left( \frac{a_iH_i}{\epsilon} \right) \right] f(a_iH_i, \lambda) \\
+ \frac{1}{2} \left[ 1 - \tanh\left( \frac{a_iH_i}{\epsilon} \right) \right] g(a_iH_i, \lambda) \right] H_k(\lambda)\rho(\lambda) d\lambda. \tag{8}
\]

the integrands can be integrated numerically
Certain Hybrid systems have a state reset conditions when a certain condition is satisfied. Example: Bouncing Ball

If $x_1 > 0$:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -g.$$  \hspace{1cm} (9)

If $x_1 = 0$:

$$\dot{x}_1^+ = x_1^-$$
$$\dot{x}_2^+ = \gamma x_2^-.$$  \hspace{1cm} (10)
Including Reset Conditions into gPC

Construct a dynamical system:

- Boundary layer around Guard in time
  \[ x^- \rightarrow h(x^-) \text{ in } \Delta t \approx \epsilon \]
- Equilibrium at:
  \[ x^- + \frac{1}{\epsilon} (h(x^-) - x^-) \]

Dynamical system:

\[ \dot{x} = x^- + \frac{1}{\epsilon} (h(x^-) - x^-) - x : \text{ if } |x| \in G(q, q') \]

Append equations to system of equations for hybrid system
Multi-modal Distributions

Different parameter values may map to different regions in output space

Multi-Element gPC is a possible approach\(^1\)

Multi-Element gPC

- Partition the random space into elements:
  \[ B_k = [a_{k,1}, b_{k,1}] \times [a_{k,2}, b_{k,2}] \times \cdots \times [a_{k,d}, b_{k,d}] \].

- \( D = \bigcup_{k=1}^{N} B_k \) such that \( B_{k_1} \cap B_{k_2} = \emptyset \) if \( k_1 \neq k_2 \)

- Perform gPC in every element using the part of the PDF in that element
  - One can generate orthogonal polynomials using the three term recurrence\(^2\): \( \pi_{i+1}(\tau) = (\tau - \alpha_i)\pi_i(\tau) - \beta_i\pi_{i-1}(\tau), \pi_0(\tau) = 1, \pi_{-1}(\tau) = 0 \).

- One can also generate collocation points for each element to give: ME-PCM.

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Result for the Bimodal Distribution

- Monte Carlo (15000 samples): Mean = $8.1 \times 10^{-3}$ and variance = 6.4137
- ME-PCM (2 elements with 9 collocation points in each element): Mean = 0 and variance = 6.4563
- Results obtained with 15000 samples of Monte Carlo comparable to 18 samples of ME-PCM.
Switching Oscillator

Use system below to test UQ approaches

\[
\begin{align*}
\ddot{x} + c\dot{x} + x + \lambda &= 0 \text{ if } x \geq 0 \\
\ddot{x} + c\dot{x} + x - \lambda &= 0 \text{ if } x \leq 0.
\end{align*}
\] (11)

Pick two cases for $\lambda$ Gaussian with:

- Case 1: mean($\lambda$) = 10, std($\lambda$) = 2
- Case 2: mean($\lambda$) = $-10$, std($\lambda$) = 2.
Dynamics of the oscillator

Case 1: $\mu(\lambda) = 10 \& \sigma(\lambda) = 2$

![Graph 1](image1.png)

Case 2: $\mu(\lambda) = -10 \& \sigma(\lambda) = 2$

![Graph 2](image2.png)
Case 1: Comparison between MC, QMC and PCM

**Oscillator Dynamics**

Why do gPC/PCM fail?

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**Outline**

- Introduction
- Challenges Posed by Hybrid Systems
- Results
- Future Work

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Uncertainty Quantification for Hybrid Dynamical Systems
Case 2: Comparison between MC, QMC and PCM

- **Mean of x**
  - MC
  - QMC
  - Collocation

- **Variance of x**
  - MC
  - QMC
  - Collocation

**Why do gPC/PCM fail?**
gPC on Case 1 & 2:

Case 1:

Why do gPC and Probabilistic Collocation fail? Case 2: gives perfect results (not shown here)
gPC/PCM on Case 1: Why does it fail?

- $x(t; \lambda)$ becomes more and more oscillatory with respect to $\lambda$ as $t \uparrow$ due to switching.
- The order of expansion of $x(t; \lambda) = a_i(t)H_i(\lambda)$ is fixed beforehand.
- This implies: for any $\epsilon > 0$, $\exists t_f$ s.t.
  $$|x(t_f; \lambda) - a_i(t_f)H_i(\lambda)| > \epsilon$$
ME-PCM provides more accurate approximation to the solution. 36 elements with 5 collocation points.
Future Work

- Compare above methods to methods for stochastic hybrid systems - particularly for reachability analysis
- Can ME-gPC/PCM handle entire classes of Hybrid systems?
- How does one locate elements for ME-gPC? - Adaptive methods?
Systems (hybrid or continuous) with large number of uncertain parameters:

- Perform gPC/ME-gPC to give a high dimensional dynamical system
- How does one do computation of extremely high dimensional systems? Possible answer: Adaptive Waveform Relaxation
- Partition ODE system using a distributed graph partitioning algorithm

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4. T. Sahai, A. Speranzon and A. Banaszuk, Hearing the Clusters of a Graph: A Distributed Algorithm, arxiv:0911.4729