Uncertainty Quantification for Hybrid Dynamical Systems

Tuhin Sahai

United Technologies Research Center

May 26, 2010

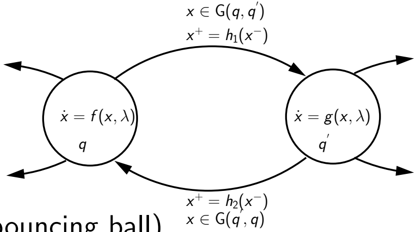
with José Miguel Pasini

Why is it important?

- Hybrid dynamical systems are a useful and common abstraction of systems with multiple modes of operation
- Each mode has a different set of governing differential equations
- Challenging for analysis: give rise to new phenomena like Zeno behavior
- Examples include bouncing ball, water tank etc

Problem Definition

The system switches from one mode to another if guard condition is satisfied,



$$\dot{x} = f(x, \lambda) : \text{if } x \in \text{Domain}(q)$$
$$\dot{x} = g(x, \lambda) : \text{if } x \in \text{Domain}(q')$$

Reset condition: $x^+ = h(x^-)$ (e.g. bouncing ball).
For simplicity consider (without loss of generality),

$$\begin{aligned} \dot{x} &= f(x, \lambda) : \text{if } x \geq 0 \\ \dot{x} &= g(x, \lambda) : \text{if } x < 0. \end{aligned} \tag{1}$$

λ is a vector of random parameters - Fast Uncertainty propagation.

Comparison of various methods

10000 Monte Carlo samples is considered as baseline.
Compared different methods

- Monte Carlo (MC)
- Quasi Monte Carlo (QMC)
- Generalized Polynomial Chaos (gPC) for hybrid systems
- Probabilistic Collocation Method (PCM) - 9th order approximation
- Multi Element Galerkin Polynomial Chaos (ME-gPC)

Proposed gPC method: New Methodology

Method 1:

$$\begin{aligned} \dot{x} &= f(x,\lambda) : \text{if } \ x \geq 0 \\ \dot{x} &= g(x,\lambda) : \text{if } \ x < 0. \end{aligned} \tag{2}$$

Define $\theta(x)$:

$$\begin{aligned} \theta(x) &= 1 : \text{if } \ x \geq 0 \\ \theta(x) &= 0 : \text{if } \ x < 0. \end{aligned} \tag{3}$$

Then,

$$\dot{x} = \theta(x)f(x,\lambda) + (1-\theta(x))g(x,\lambda) \tag{4}$$

Expand $x(t,\lambda) = a_i(t)H_i(\lambda)$ (Einstein notation).

Proposed gPC method: New Methodology

$$\dot{a}_i H_i = \theta(a_i H_i) f(a_i H_i, \lambda) + (1 - \theta(a_i H_i)) g(a_i H_i, \lambda) \quad (5)$$

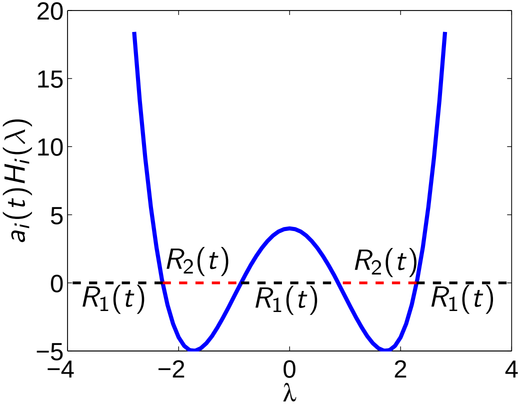
Multiply above relation by $\rho(\lambda) H_k(\lambda)$ and integrate,

$$\dot{a}_k = \int_{R_1(t)} f(a_i H_i, \lambda) \rho(\lambda) H_k(\lambda) d\lambda + \int_{R_2(t)} g(a_i H_i, \lambda) \rho(\lambda) H_k(\lambda) d\lambda \quad (6)$$

$R_1(t) = \{\lambda : a_i H_i \geq 0\}$ and $R_2(t) = R - R_1(t)$

$R_1(t)$ can be computed by looking at the zeros of $a_i(t) H_i(\lambda)$

Computing $R_1(t)$ and $R_2(t)$



Zeros of the polynomial give boundaries of region (1D)

Alternative gPC approach

Method 2:

$$\dot{x} = \frac{1}{2}\left[1 + \tanh(\frac{x}{\epsilon})\right] f(x,\lambda) + \frac{1}{2}\left[1 - \tanh(\frac{x}{\epsilon})\right] g(x,\lambda)\,. \qquad (7)$$

Expand $x(t,\lambda) = a_i(t)H_i(\lambda)$ to give,

$$\begin{aligned}\dot{a}_k \quad &= \quad \int_R \left[\frac{1}{2}\left[1 + \tanh(\frac{a_i H_i}{\epsilon})\right] f(a_i H_i, \lambda)\right. \\ &\left. + \frac{1}{2}\left[1 - \tanh(\frac{a_i H_i}{\epsilon})\right] g(a_i H_i, \lambda)\right] H_k(\lambda)\rho(\lambda) d\lambda\,. \qquad (8)\end{aligned}$$

the integrands can be integrated numerically

Including Reset Conditions into gPC

Certain Hybrid systems have a state reset conditions when a certain condition is satisfied. Example: Bouncing Ball
If $x_1 > 0$:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -g. \end{aligned} \tag{9}$$

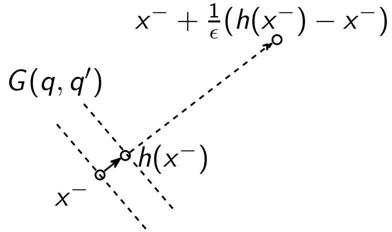
If $x_1 = 0$:

$$\begin{aligned} \dot{x}_1^+ &= x_1^- \\ \dot{x}_2^+ &= \gamma x_2^- . \end{aligned} \tag{10}$$

Including Reset Conditions into gPC

Construct a dynamical system:

- Boundary layer around Guard in time
- $x^- \to h(x^-)$ in $\Delta t \approx \epsilon$
- Equilibrium at: $x^- + \frac{1}{\epsilon}(h(x^-) - x^-)$



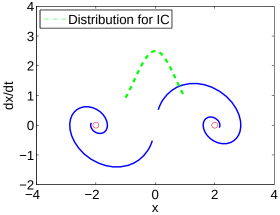
Dynamical system:

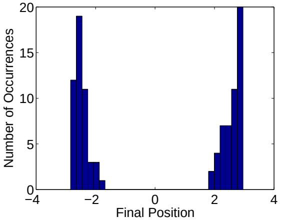
$$\dot{x} = x^- + \frac{1}{\epsilon}(h(x^-) - x^-) - x : \text{if } |x| \in G(q, q')$$

Append equations to system of equations for hybrid system

Multi-modal Distributions

Different parameter values may map to different regions in output space





Multi-Element gPC is a possible approach[1]

[1] X. Wan and G. E. Karniadakis, An adaptive multi-element generalized polynomial chaos method for stochastic differential equations, 2005.

Multi-Element gPC

- Partition the random space into elements:
  $B_k = [a_{k,1}, b_{k,1}) \times [a_{k,2}, b_{k,2}) \times \cdots \times [a_{k,d}, b_{k,d}]$.
- $D = \cup_{k=1}^{N} B_k$ such that $B_{k_1} \cap B_{k_2} = \emptyset$ if $k_1 \neq k_2$
- Perform gPC in every element using the part of the PDF in that element
    - One can generate orthogonal polynomials using the three term recurrence[2]: $\pi_{i+1}(\tau) = (\tau - \alpha_i)\pi_i(\tau) - \beta_i \pi_{i-1}(\tau)$, $\pi_0(\tau) = 1$, $\pi_{-1}(\tau) = 0$.
- One can also generate collocation points for each element to give: ME-PCM.

[2] X. Wan and G. E. Karniadakis, An adaptive multi-element generalized polynomial chaos method for stochastic differential equations, 2005.

Result for the Bimodal Distribution

- Monte Carlo (15000 samples): Mean $= 8.1 \times 10^{-3}$ and variance $= 6.4137$
- ME-PCM (2 elements with 9 collocation points in each element): Mean $= 0$ and variance $= 6.4563$
- Results obtained with 15000 samples of Monte Carlo comparable to 18 samples of ME-PCM.

Switching Oscillator

Use system below to test UQ approaches

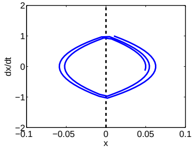
$$\begin{aligned} \ddot{x} + c\dot{x} + x + \lambda &= 0 \text{ if } x \geq 0 \\ \ddot{x} + c\dot{x} + x - \lambda &= 0 \text{ if } x \leq 0. \end{aligned} \tag{11}$$

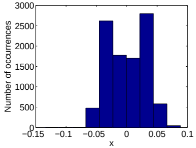
Pick two cases for λ Gaussian with:

- Case 1: $\text{mean}(\lambda) = 10, \text{std}(\lambda) = 2$
- Case 2: $\text{mean}(\lambda) = -10, \text{std}(\lambda) = 2.$

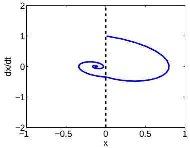
Dynamics of the oscillator

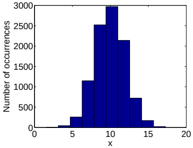
Case 1: $\mu(\lambda) = 10$ & $\sigma(\lambda) = 2$



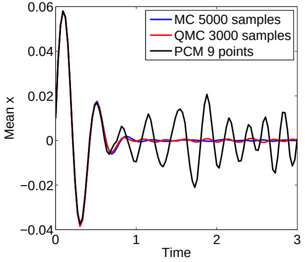


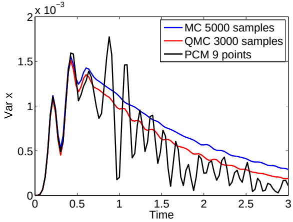
Case 2: $\mu(\lambda) = -10$ & $\sigma(\lambda) = 2$



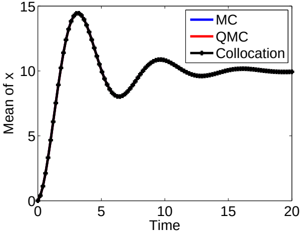


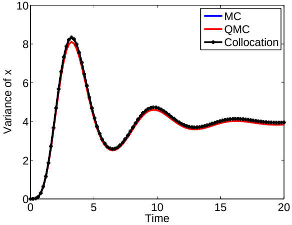
Case 1: Comparison between MC, QMC and PCM





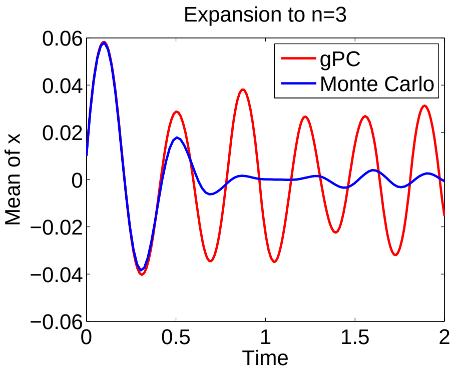
Case 2: Comparison between MC, QMC and PCM





gPC on Case 1 & 2:

Case 1:

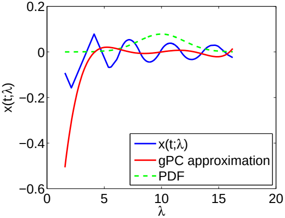


Why do gPC and Probabilistic Collocation fail?
Case 2: gives perfect results (not shown here)

gPC/PCM on Case 1: Why does it fail?

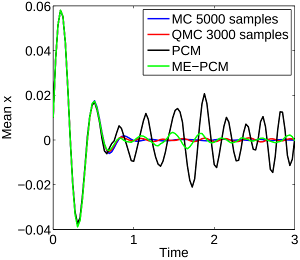
- $x(t;\lambda)$ becomes more and more oscillatory with respect to λ as $t \uparrow$ due to switching.
- The order of expansion of $x(t;\lambda) = a_i(t)H_i(\lambda)$ is fixed beforehand
- This implies: for any $\epsilon > 0$, $\exists t_f$ s.t. $|x(t_f;\lambda) - a_i(t_f)H_i(\lambda)| > \epsilon$

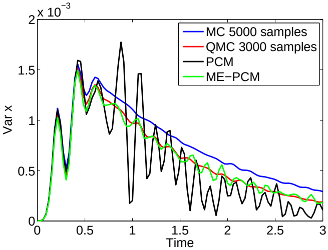
(Loading xvst.avi)



Performance of ME-PCM

ME-PCM provides more accurate approximation to the solution.
36 elements with 5 collocation points.





Future Work

- Compare above methods to methods for stochastic hybrid systems - particularly for reachability analysis
- Can ME-gPC/PCM handle entire classes of Hybrid systems?
- How does one locate elements for ME-gPC? - Adaptive methods?

UQ for High-Dimensional Systems

Systems (hybrid or continuous) with large number of uncertain parameters:

- Perform gPC/ME-gPC to give a high dimensional dynamical system
- How does one do computation of extremely high dimensional systems? Possible answer: Adaptive Waveform Relaxation [3]
- Partition ODE system using a distributed graph partitioning algorithm [4]

[3] S. Klus, T. Sahai, C. Liu and M. Dellnitz, An efficient algorithm for the parallel solution of high-dimensional differential equations,arxiv:1003.5238

[4] T. Sahai, A. Speranzon and A. Banaszuk, Hearing the Clusters of a Graph: A Distributed Algorithm,arxiv:0911.4729