Approximation of an Optimal Control Problem for Stochastic PDEs

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Outline

Outline

Introduction

Stochastic PDEs

Stochastic Control Problems

Numerical Simulations

Model Problem

Constraint Equation:

$$-\nabla \cdot (a(x,\omega)\nabla u(x,\omega)) = f(x,\omega) \quad \text{in } D,$$

$$(a(x,\omega)\nabla u(x,\omega)) \cdot \mathbf{n} = p(x) \quad \text{on } \partial D.$$

- u is the pressure, a is the permeability field, (the randomness enters through the unknown properties of the porous media)
- f is the source term, $\mathbf{n}$ is a unit outward normal to the boundary, and p is flexible input data (the control).

Objective Functional:

$$\mathcal{J}(u,p) = E\left(\frac{1}{2}\int_D |u-U|^2\, dx + \frac{\beta}{2}\int_{\partial D} |p|^2\, dx\right).$$

Karhunen-Loève (K-L) expansions

- K-L Expansion:

$$a(x,\omega) = \bar{a}(x) + \sum_{n\geq 1} \sqrt{\lambda_n}\phi_n(x)X_n(\omega),$$

where $\bar{a}(x) = Ea(x,\omega)$, $EX_n(\omega) = 0$,
$E(X_n(\omega)X_m(\omega)) = \delta_{nm}$, and $(\lambda_n, \phi_n(x))$ are solutions to

$$\int_D C(x_1,x_2)\phi_n(x_1)\;dx_1 = \lambda_n\phi_n(x_2),$$

where continuous covariance function

$$C(x_1,x_2) = E(a(x_1,\omega)a(x_2,\omega)) - Ea(x_1,\omega)Ea(x_2,\omega).$$

Outline

Assumption

- $(\Omega, \mathcal{F}, P)$: a probability space, where Ω is a set of outcomes, $\mathcal{F}$ is a σ-algebra of events, and $P : \mathcal{F} \to [0, 1]$ is a probability measure.
- Assume that there are $m, M \in (0, \infty)$ such that

$$m \leq a(x, \omega) \leq M \qquad \text{a.e. } (x, \omega) \in D \times \Omega.$$

- Assume that

$$E \int_D |f|^2 \, dx = \int_\Omega \int_D |f|^2 \, dx dP(\omega) < \infty.$$

Function Spaces

- Stochastic Sobolev space:

$$L^2(\Omega; H^r(D)) = \{v : D \times \Omega \to \mathbb{R} \mid \ \|v\|_{L^2(\Omega;H^r(D))} < \infty\},$$

where $\|v\|^2_{L^2(\Omega;H^r(D))} = \int_\Omega \|v\|^2_{H^r(D)}\, dP = E\|v\|^2_{H^r(D)}$.
For example,

$$L^2(\Omega; H^1_0(D)) = \{v : D \times \Omega \to \mathbb{R} \mid E \int_D |\nabla v|^2\, dx < \infty\}.$$

Notations

- Notations:

$$\mathcal{H}^r(D) = L^2(\Omega; H^r(D)),\ \mathcal{L}^2(D) = L^2(\Omega; L^2(D)),$$

$$\mathcal{H}^1(D) = \{v \in \mathcal{L}^2(D) \mid E\|v\|^2_{H^1(D)} < \infty\},$$

$$\mathcal{H}^1_0(D) = \{v \in \mathcal{H}^1(D) \mid v = 0 \quad \text{on } \partial D\ a.s.\},$$

$$b[u, v] = E\int_D a\nabla u \cdot \nabla v\ dx,$$

$$[u, v] = E\int_D uv\ dx,$$

and

$$[u, v]_{\partial D} = E\int_{\partial D} uv\ dx.$$

The Solution of a Stochastic PDE

- Model Equation:

$$-\nabla \cdot (a(x,\omega)\nabla u(x,\omega)) = f(x,\omega) \quad \text{in } D,$$

$$(a(x,\omega)\nabla u(x,\omega)) \cdot \mathbf{n} = p(x) \quad \text{on } \partial D.$$

- Weak formulation: seek $u \in \mathcal{H}^1(D)$ such that

$$b[u,v] = [f,v] + [p,v]_{\partial D} \quad \forall v \in \mathcal{H}^1(D).$$

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Theorem

Let $f \in \mathcal{L}^2(D)$ and $p \in L^2(\partial D)$. Then there exists a unique weak solution in $\mathcal{H}^1(D)$, provided that $E\int_D f\ dx + \int_{\partial D} p\ dx = 0$.

Assumption

- In realistic models, the source of randomness can be expressed by a finite number of random variables that are mutually independent.
- Whenever we apply some numerical method to solve model equations, we assume that stochastic functions are finite expansions.
- Assume that

$$m \leq a(x,\omega) = \bar{a}(x) + \sum_{n=1}^{N} \sqrt{\lambda_n}\phi_n(x)X_n(\omega) \leq M.$$

- $X_n(\Omega) \equiv \Gamma_n \subset \mathbb{R}$ is a bounded interval for $n = 1, 2, \cdots, N$.
- X_n has a probability density function $\rho_n : \Gamma_n \to \mathbb{R}^+$ for $n = 1, 2, \cdots, N$.
- We use the joint probability density function $\rho(y)$ of $(X_1, X_2, \cdots, X_N)$ for any $y \in \Gamma \equiv \prod_{n=1}^{N} \Gamma_n \subset \mathbb{R}^N$.

Deterministic Equivalent Formulation

- Weak Formulation:

$$\int_\Gamma \rho(y) \int_D a(x,y)\nabla u(x,y)\cdot\nabla v(x,y)\ dxdy$$
$$= \int_\Gamma \rho(y) \int_D f(x,y)v(x,y)\ dxdy$$
$$+ \int_\Gamma \rho(y) \int_{\partial D} p(x)v(x,y)\ dxdy \quad \forall v \in \mathcal{H}^1(D).$$

- Strong Formulation:

$$-\nabla\cdot(a(x,y)\nabla u(x,y)) = f(x,y) \quad \forall (x,y) \in D\times\Gamma,$$

$$(a(x,y)\nabla u(x,y))\cdot\mathbf{n} = p(x) \quad \forall (x,y) \in \partial D\times\Gamma.$$

Finite Element Spaces on D

- $X^h \subset H^1(D)$ is a family of finite element approximation spaces that consist of piecewise linear continuous functions defined over a family of regular triangulations of D with a maximum grid size parameter $h > 0$.

- Assume that X^h satisfy the following approximation property: (i) for all $\phi \in H^2(D)$, there exists $C > 0$ such that

$$\inf_{\phi^h \in X^h} \|\phi - \phi^h\|_{H^1(D)} \leq Ch\|\phi\|_{H^2(D)}.$$

(ii) for all $\phi \in H^1(D)$, there exists $C > 0$ such that

$$\inf_{\phi^h \in G^h} \|\phi - \phi^h\|_{L^2(D)} \leq Ch\|\phi\|_{H^1(D)}.$$

Finite Element Spaces on Γ

- Partition Γ into a finite number of disjoint $\mathbb{R}^N$ boxes B_i^N: For a finite index set $\mathcal{I}$,

$$\Gamma = \bigcup_{i \in I} B_i^N = \bigcup_{i \in \mathcal{I}} \prod_{j=1}^{N} (a_i^j,\ b_i^j),$$

  where $B_k^N \cap B_l^N = \emptyset$ for $k \neq l \in \mathcal{I}$ and $(a_i^j,\ b_i^j) \subset \Gamma_j$.
- Maximum grid size parameter $\delta > 0$:

$$\delta = \max\{|b_i^j - a_i^j|/2 : 1 \leq j \leq N \text{ and } i \in \mathcal{I}\}.$$

- $Y^\delta \subset L^2(\Gamma)$ is the finite element approximation space of piecewise polynomials with degree at most p_j on each direction y_j.

Finite Element Spaces on $D \times \Gamma$

- Finite element spaces $V^{h\delta}$ on $D \times \Gamma$.
- $v^{h\delta} \in V^{h\delta} \Rightarrow v^{h\delta} \in \text{span}(\phi^h \psi^\delta \ : \ \phi^h(x) \in X^h, \psi^\delta(y) \in Y^\delta)$
- Weak Formulation:
  Seek $u \in \mathcal{H}^1(D)$ such that for all $v \in \mathcal{H}^1(D)$,

$$b[u, v] = [f, v] + [p, v]_{\partial D}.$$

- Finite Element Weak Formulation:
  Find $u^{h\delta} \in V^{h\delta}$ such that for all $v^{h\delta} \in V^{h\delta}$,

$$b[u^{h\delta}, v^{h\delta}] = [f, v^{h\delta}] + [p, v^{h\delta}]_{\partial D}.$$

Error Estimates

Theorem

Let $f \in C^{p+1}(\Gamma; H^1(D))$, $p \in H^{1/2}(D)$, and u be the solution of the weak form, and let $u^{h\delta}$ be the finite element solution of the finite element weak form. Then there exists $C > 0$ such that

$$\|u - u^{h\delta}\|_{\mathcal{H}^1(D)} \leq C \left(h + \delta^\gamma) K (\|f\|_{\mathcal{L}^2(D)} + \|p\|_{H^{1/2}(D)} \right),$$

where $K = \sum_{j=1}^{N} \max \left\{ 1, \|\phi_j\|_{L^\infty(D)}^{p_j+1}, \sum_{k=1}^{p_j+1} \|\phi_j\|_{L^\infty(D)}^{p_j+1-k} \|\partial_{y_j}^k f\|_{\mathcal{L}^2(D)} \right\}.$

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Optimal Control Problem

Minimize

$$\mathcal{J}(u,p) = E\left(\frac{1}{2}\int_D |u-U|^2 \, dx + \frac{\beta}{2}\int_{\partial D} |p|^2 \, dx\right)$$

subject to

$$-\nabla \cdot (a(x,\omega)\nabla u(x,\omega)) = f(x,\omega) \quad \text{in } D,$$

$$(a(x,\omega)\nabla u(x,\omega)) \cdot \mathbf{n} = p(x) \quad \text{on } \partial D.$$

Existence of an Optimal Solution

- Admissibility Set:

$$\mathcal{U}_{ad} = \Big\{ (u,p) \in \mathcal{H}^1(D) \times H^{1/2}(\partial D) \text{ such that} \\ \text{the weak form satisfied and } \ \mathcal{J}(u,p) < \infty \Big\}.$$

- Optimal Solution:
  $(\hat{u}, \hat{p}) \in \mathcal{U}_{ad}$ is called an *optimal solution* of $\mathcal{J}(u,p)$ if $\mathcal{J}(\hat{u}, \hat{p}) \leq \mathcal{J}(u,p)$ for all $(u,p) \in \mathcal{U}_{ad}$ satisfying that $\|u - \hat{u}\|_{\mathcal{H}^1(D)} + \|p - \hat{p}\|_{H^{1/2}(\partial D)} \leq \epsilon$ for some $\epsilon > 0$.

Existence of an Optimal Solution

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Theorem

There is an optimal solution $(\hat{u}, \hat{p}) \in \mathcal{U}_{ad}$.

Lagrange's Method

- Minimization Problem:

$$\min \mathcal{J}(u,p) \text{ subject to } M(u,p)=0,$$

where $M(u,f) = b[u,v] - [f,v] - [p,v]_{\partial D}$.

- Lagrangian:

$$\mathcal{L}(u,p,\xi) = \mathcal{J}(u,p) - b[u,\xi] + [f,\xi] + [p,\xi]_{\partial D}.$$

The abstract minimization problem

- Let G, X, and Y be reflexive Banach Spaces whose norm are denoted by $\|\cdot\|_G, \|\cdot\|_X$, and $\|\cdot\|_Y$ and whose dual spaces are denoted by G^*, X^*, and Y^*, respectively.
- Let Θ be the control set that is a closed convex subset of G.
- We assume that the functional to be minimized takes the form

$$\mathcal{J}(v, z) = \lambda\mathcal{F}(v) + \lambda\mathcal{E}(z) \quad \forall(v, z) \in X \times \Theta,$$

where $\mathcal{F}$ is a functional on X, $\mathcal{E}$ is a functional on Θ, and λ is a given parameter that is assumed to belong to a compact interval $\Lambda \subset \mathbb{R}_+$.

The abstract minimization problem

- We define the function $M : X \times \Theta \to X$ for the constraint equation $M(v, z) = 0$ as follows:

$$M(v, z) = v + \lambda T N(v) + \lambda T K(z) \quad \forall (v, z) \in X \times \Theta,$$

  where $N : X \to Y$ is a differentiable map, $K : \Theta \to Y$ is a bounded linear operator, $T : Y \to X$ is a bounded linear operator, and $\lambda \in \Lambda$.
- With these definitions, we now consider the following constrained minimization problem:

$$\min_{(v,z)\in X\times\Theta} \mathcal{J}(v, z) \quad \text{subject to} \quad M(v, z) = 0.$$

Hypotheses concerning the abstract minimization problem

- The set of hypotheses needed to justify the use of the Lagrange multiplier rule and to derive an optimality system from which optimal states and controls can be determined is given by
  - (HE1) for each $z \in \Theta$, $v \mapsto \mathcal{J}(v,z)$ and $v \mapsto M(v,z)$ are Fréchet differentiable;
  - (HE2) $z \mapsto \mathcal{E}(z)$ is convex;
  - (HE3) for $v \in X$, $N'(v)$ maps from X into $Z \hookrightarrow\hookrightarrow Y$, where N' denotes the Fréchet derivative of N.

The abstract minimization problem

Theorem(Tikomirov)

Let $\lambda \in \Lambda$ be given. Assume that there exists an optimal solution (u, f) of the minimization problem in $X \times G$, that $(HE1) - (HE3)$ hold, and that the mapping $z \mapsto \mathcal{E}(z)$ is Frechet differentiable on G. Then there exists a $k \in \mathbb{R}$ and a $\mu \in X^*$, not both equal to zero, such that

$$k\langle \mathcal{J}_u(u, f), w\rangle - \langle \mu, (I + \lambda T N'(u)) \cdot w\rangle = 0 \quad \forall w \in X$$

and

$$k\langle \mathcal{E}'(f), z\rangle - \langle \mu, TKz\rangle = 0 \quad \forall z \in G.$$

Existence of Lagrange multiplier

- We prove the existence of a Lagrange multiplier for our minimization problem.
- The Lagrange multiplier rule may be used to convert the constrained minimization problem into an unconstrained one.
- Let $X = \mathcal{H}^1(D)$, $Y = \mathcal{H}^{-1}(D)$, $G = \mathcal{L}^2(\partial D)$, and $Z = \{0\}$.
- Then clearly we have $Z \hookrightarrow\hookrightarrow Y$.
- For the time being, we assume that the admissible set Θ for the control p is a closed, convex subset of G.
- Let the continuous linear operator $T \in \mathcal{L}(Y; X)$ as follows: for $g \in Y$, $Tg = u \in X$ is the unique solution of

$$b[u, v] = [g, v] \quad \forall v \in X.$$

Existence of Lagrange multiplier

- We define the (differentiable) mapping $N : X \to Y$ by

$$N(u) = -f \quad \forall u \in X$$

or, equivalently,

$$\langle N(u), v \rangle = \langle -f, v \rangle \quad \forall v \in X$$

and define $K : G \to Y$ as the injection mapping

$$\langle Kp, \eta \rangle = -\langle p, \eta \rangle \quad \forall p \in G \quad \forall \eta \in X.$$

- Then it is clear that $b[u,v]_D - [f,v]_D = [p,v]_{\partial D}$ can be expressed as $u + TN(u) + TKp = 0$.
-

$$\mathcal{F}(u) = E\left(\frac{1}{2}\int_D |u-U|^2 \, dx\right) \text{ and } \mathcal{E}(p) = E\left(\frac{\beta}{2}\int_{\partial D} |p|^2 \, dx\right)$$

Verifying the hypotheses

- First, notice that $(HE1)$ is obvious.
- Second, $(HE2)$ holds because $f \mapsto \mathcal{E}(p) = \frac{\beta}{2}\|p\|^2_{\mathcal{L}^2(\partial D)}$ is convex.
- Third, because $N'(u) \cdot v = 0 \in Z \hookrightarrow\hookrightarrow Y$ for $\forall u, v \in X$, $(HE3)$ holds.
- The Lagrangian is given by

$$\mathcal{L}(u, p, \xi) = \mathcal{J}(u, p) - b[u, \xi]_D + [f, \xi]_D + [p, \xi]_{\partial D}$$

  for $\forall (u, f, \xi) \in X \times G \times X$.
- By the Theorem(Tikomirov), there exists $\xi = T^*\mu \in X$ such that

$$\xi - T^*\mathcal{F}'(u) = 0$$

  and

$$\mathcal{L}(u, p, \xi) \le \mathcal{L}(u, z, \xi) \quad \forall z \in \Theta$$

Stochastic Optimality System of Equations

- Stochastic Optimality System of Equations:

$$b[u,v] = [f,v] + [p,v]_{\partial D} \quad \forall v \in \mathcal{H}^1(D),$$

$$b[\xi,\zeta] = [u-U,\zeta] \quad \forall \zeta \in \mathcal{H}^1(D),$$

and

$$[\beta p + \xi, z]_{\partial D} = 0 \quad \forall z \in L^2(\partial D).$$

Theorem

Let $(u,p) \in \mathcal{H}^1(D) \times \mathcal{L}^2(\partial D)$ be an optimal solution of the minimization problem. Then there exists $\xi \in \mathcal{H}^1(D)$ such that the second and third equations of the above equations hold.

Reduced Stochastic Optimality System of Equations

- Reduced Stochastic Optimality System of Equations:

$$b[u,v] = [f,v] - \beta^{-1}[\xi_{\partial D}, v]_{\partial D} \quad \forall v \in \mathcal{H}^1(D)$$

and

$$b[\xi,\zeta] = [u-U,\zeta] \quad \forall \zeta \in \mathcal{H}^1(D),$$

where $\xi_{\partial D}(x) = \int_\Gamma \xi(x,y)\, dy$.

Brezzi-Rappaz-Raviart (BRR) theory

- Idea: Find error estimates for non-linear problems by using error estimates for linear problems.
- Non-linear Problems: Seek $\psi \in \mathcal{X}$ such that

$$\psi + \mathcal{T}\mathcal{G}(\psi) = 0,$$

  where $\mathcal{T} \in \mathcal{L}(\mathcal{Y}; \mathcal{X})$, $\mathcal{G}$ is a C^2 mapping from $\mathcal{X}$ into $\mathcal{Y}$, and $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces.
- Assumption: There exists a Banach space $\mathcal{Z} \subset \mathcal{Y}$, with continuous imbedding, such that

$$\mathcal{G}_\psi(\psi) \in \mathcal{L}(\mathcal{X}; \mathcal{Z}) \quad \forall \psi \in \mathcal{X}.$$

Brezzi-Rappaz-Raviart (BRR) theory

- Approximations: For $\mathcal{X}^h \subset \mathcal{X}$ and for $\mathcal{T}^h \in \mathcal{L}(\mathcal{Y}; \mathcal{X}^h)$, seek $\psi^h \in \mathcal{X}^h$ such that

$$\psi^h + \mathcal{T}^h \mathcal{G}(\psi^h) = 0.$$

- Assumption:

$$\lim_{h \to 0} \|(\mathcal{T}^h - \mathcal{T})\omega\|_{\mathcal{X}} = 0 \quad \forall \omega \in \mathcal{Y}$$

and

$$\lim_{h \to 0} \|\mathcal{T}^h - \mathcal{T}\|_{\mathcal{L}(\mathcal{Z};\mathcal{X})} = 0.$$

Recasting the Optimality System into the BRR Framework

- Set $\mathcal{X} = \mathcal{H}^1(D) \times \mathcal{H}^1(D)$ and $\mathcal{Y} = \mathcal{H}^{-1}(D) \times H^{1/2}(\partial D) \times \mathcal{H}^{-1}(D)$.
- Define the linear operator $\mathcal{T} \in \mathcal{L}(\mathcal{Y}; \mathcal{X})$ as follows:

$$(u, \xi) = \mathcal{T}(\tilde{\sigma}, \tilde{\varsigma}, \tilde{\tau})$$

if and only if

$$b[u, v] = [\tilde{\varsigma}, v] + [\tilde{\tau}, v]_{\partial D} \quad \forall v \in \mathcal{H}^1(D)$$

and

$$b[\xi, \zeta] = [\tilde{\sigma}, \zeta] \quad \forall \zeta \in \mathcal{H}^1(D),$$

Recasting the Optimality System into the BRR Framework

- Define $\mathcal{G} : \mathcal{X} \to \mathcal{Y}$ by

$$\mathcal{G}(u, \xi) = (-f, \beta^{-1}\xi_{\partial D}, U - u).$$

- Then it is clear that the optimality system can be written as

$$(u, \xi) + \mathcal{T}(\mathcal{G}(u, \xi)) = 0.$$

- Hence, the optimality system is recast into the form of BRR.

Recasting the Optimality System into the BRR Framework

- Set $\mathcal{X}^{h\delta} = V^{h\delta} \times V^{h\delta}$.
- Define the discrete operator $\mathcal{T}^{h\delta} \in \mathcal{L}(\mathcal{Y}; \mathcal{X}^{h\delta})$ as follows:

$$(u^{h\delta}, \xi^{h\delta}) = \mathcal{T}^{h\delta}(\tilde{\sigma}, \tilde{\varsigma}, \tilde{\tau})$$

if and only if

$$b[u^{h\delta}, v^{h\delta}] = [\tilde{\varsigma}, v^{h\delta}] + [\tilde{\tau}, v^{h\delta}]_{\partial D} \quad \forall v^{h\delta} \in V^{h\delta}$$

and

$$b[\xi^{h\delta}, \zeta^{h\delta}] = [\tilde{\sigma}, \zeta^{h\delta}] \quad \forall \zeta^{h\delta} \in V^{h\delta}.$$

Recasting the Optimality System into the BRR Framework

- Then it is clear that the discrete optimality system,

$$b[u^{h\delta}, v^{h\delta}] = [\tilde{\varsigma}, v^{h\delta}] + [\tilde{\tau}, v^{h\delta}]_{\partial D} \quad \forall v^{h\delta} \in V^{h\delta}$$

  and

$$b[\xi^{h\delta}, \zeta^{h\delta}] = [\tilde{\sigma}, \zeta^{h\delta}] \quad \forall \zeta^{h\delta} \in V^{h\delta}.$$

  can be written as

$$(u^{h\delta}, \xi^{h\delta}) + \mathcal{T}^{h\delta}(\mathcal{G}(u^{h\delta}, \xi^{h\delta})) = 0.$$

- Hence, the discrete optimality system is recast into the form of BRR.

Verifying All Assumptions in BRR Theorem

Lemma

$D\mathcal{G}(u,\xi) \in \mathcal{L}(\mathcal{X};\mathcal{Z})$ $for\ all$ $(u,\xi) \in \mathcal{X}$, where $\mathcal{Z} = \mathcal{L}^2(D) \times H^1(D) \times \mathcal{L}^2(D) \hookrightarrow \mathcal{Y}$

Lemma

$\mathcal{G}$ is twice continuously differentiable and $D^2\mathcal{G}$ is bounded on all bounded sets of $\mathcal{X}$.

Lemma

For any $(\tilde{\sigma}, \tilde{\varsigma}, \tilde{\tau}) \in \mathcal{Y}$, $\|(\mathcal{T} - \mathcal{T}^{h\delta})(\tilde{\sigma}, \tilde{\varsigma}, \tilde{\tau})\|_{\mathcal{X}} \to 0$ as $h, \delta \to 0$.

Lemma

$\|\mathcal{T} - \mathcal{T}^{h\delta}\|_{\mathcal{L}(\mathcal{Z},\mathcal{X})} \to 0$ as $h, \delta \to 0$.

Error Estimates

Theorem

Assume that $U \in \mathcal{H}^1(D)$. Let $(u, \xi) \in \mathcal{H}^1(D) \times \mathcal{H}^1(D)$ be the solution of the reduced optimality system. Let $(u^{h\delta}, \xi^{h\delta}) \in V^{h\delta} \times V^{h\delta}$ be the solution of the reduced discrete optimality system. Then we have

$$\|u - u^{h\delta}\|_{\mathcal{H}^1(D)} + \|\xi - \xi^{h\delta}\|_{\mathcal{H}^1(D)} \to 0 \text{ as } h, \delta \to 0.$$

Moreover, there exists $C > 0$ such that

$$\begin{aligned} &\|u - u^{h\delta}\|^2_{\mathcal{H}^1(D)} + \|\xi - \xi^{h\delta}\|^2_{\mathcal{H}^1(D)} \\ &\quad \le C(h^2 + \delta^{2\gamma})K\left(\|f\|^2_{\mathcal{L}^2(D)} + \|p\|^2_{H^{1/2}(D)} + \|u - U\|^2_{\mathcal{L}^2(D)}\right). \end{aligned}$$

Outline

The Optimality System of Equations

Stochastic Optimality System of Equations:

$$b[u,v] = [f,v] + [p,v]_{\partial D} \quad \forall v \in \mathcal{H}^1(D),$$

$$b[\xi,\zeta] = [u-U,\zeta] \quad \forall \zeta \in \mathcal{H}^1(D),$$

and

$$[\beta p + \xi, z]_{\partial D} = 0 \quad \forall z \in L^2(\partial D).$$

Reduced Stochastic Optimality System of Equations:

$$b[u,v] = [f,v] - \beta^{-1}[\xi_{\partial D}, v]_{\partial D} \quad \forall v \in \mathcal{H}^1(D)$$

and

$$b[\xi,\zeta] = [u-U,\zeta] \quad \forall \zeta \in \mathcal{H}^1(D),$$

Computational Methods and Algorithms

- It is my ongoing Project
- Monte Carlo
- Quasi Monte Carlo
- Multilevel Monte Carlo
- Multigrid Monte Carlo
- Sparse Grid
- Anisotropic Sparse Grid
- ROM basis method
- ...

Thanks for your attention !