Multiscale Methods for the Valuation of American Options with Stochastic Volatility

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Motivation, Problems and Wish List

- Valuation of American options ⇝ parabolic boundary value problem in variational form with free boundary (optimal strike prize of option)
- Stochastic volatility
- (Pointwise partial derivatives of solution up to order 2 ⇝ discretizations of high order)
- Numerical solution methods with multigrid efficiency

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Problem

Valuation of American Put Option using Black–Scholes' equation

$S = S(t,\omega)$ stock price $dS(t,\cdot) = \mu S(t\,\cdot)\,dt + \sqrt{\sigma(t,\cdot)}\,S(t,\cdot)\,dW_1(t,\cdot)$

$\sigma = \sigma(t,\omega)$ volatility $d\sigma(t,\cdot) = \kappa(\theta - \sigma(t,\cdot))\,dt + \xi\sqrt{\sigma(t,\cdot)}\,dW_2(t,\cdot)$

[Heston 1993]: variance σ is Cox-Ingersoll-Ross (square root diffusion) process [1985]

μ drift κ mean reversion level θ mean reversion rate ξ volatility of CIR process

$W_1(t,\cdot), W_2(t,\cdot)$ Wiener processes correlated by $\varrho \in [-1,1]$

$V = V(t, S(t,\cdot), \sigma(t,\cdot))$ value of American put option

(Ultimate) Goal: Accurate computation of V and of Greek Letters (risk parameters)

Delta $:= \frac{\partial V}{\partial S}$ Gamma $:= \frac{\partial^2 V}{\partial S^2}$ Theta $:= \frac{\partial V}{\partial t}$

Itô calculus $\rightsquigarrow$

$$\mathcal{L}\,V := \frac{\partial V}{\partial t} + \frac{1}{2}\left(S^2\sigma\frac{\partial^2 V}{\partial S^2} + 2\varrho\xi\,\sigma\,S\,\frac{\partial^2 V}{\partial S\,\partial\sigma} + \xi^2\sigma\,\frac{\partial^2 V}{\partial\sigma^2}\right) + rS\,\frac{\partial V}{\partial S} + (\kappa(\theta-\sigma) - \lambda\sigma)\frac{\partial V}{\partial\sigma} - rV = 0$$

r interest rate λ prize of volatility risk

$\rightsquigarrow$ for every $t, \sigma > 0$ there exists $S_f > 0$ ("free boundary") such that early exercise of option is optimal

$\rightsquigarrow$ immediate exercise is optimal if $S \le S_f$ or not optimal if $S > S_f$

Problem (Cont'd)

$V = V(t, S(t,\cdot), \sigma(t,\cdot))$ value of American put option on $\Omega_{\mathcal{L}}^{\infty} := [0,T] \times [0,\infty) \times [0,\infty)$

(Ultimate) Goal: Accurate computation of V and of Greek Letters (risk parameters)

$$\text{Delta} = \frac{\partial V}{\partial S} \qquad \text{Gamma} = \frac{\partial^2 V}{\partial S^2} \qquad \text{Theta} = \frac{\partial V}{\partial t}$$

$$\begin{array}{lll} \mathcal{L}\,V = 0 & \text{and } V(t,S,\sigma) \geq \max\{K-S,0\} =: \mathcal{H}(S) & \text{for } S > S_f \quad \sigma > 0 \quad 0 \leq t < T \\ \mathcal{L}\,V \leq 0 & \text{and } V(t,S,\sigma) = \mathcal{H}(S) & \text{for } S \leq S_f \end{array}$$

$+$ boundary conditions $V(t,S,0) = Ke^{-r(T-t)} - S \qquad V(t,S,\infty) = Ke^{r(T-t)} \ldots$

$\rightsquigarrow$ Parabolic boundary value problem with free boundary

$\rightsquigarrow$ Linear Complementarity Problem

$$\begin{array}{rcl} \mathcal{L}V\ (V-\mathcal{H}) & = & 0 \\ \mathcal{L}V & \leq & 0 \\ V & \geq & \mathcal{H} \end{array}$$

$+$ boundary conditions $+$ end condition $V(T,S,\sigma) = \mathcal{H}(S)$

Solution Scheme and Results

Free parabolic boundary value problem as obstacle problem

$\rightsquigarrow$ variational inequality of elliptic type (in semi–discrete form)

$$\text{find } u \in \mathcal{K} : \quad a(u, v-u) \geq f(v-u)$$

$$\text{for all } v \in \mathcal{K} := \{v \in H_0^1(\Omega) : \; v(x) \leq g(x) \text{ for all } x \in \Omega\}$$

Monotone multigrid methods (MMG) [Hackbusch, Mittelmann 1983, Mandel 1984, Hoppe 1987, Kornhuber 1994, 1997, Krause 2001, Oosterlee 2003, Reisinger, Wittum 2004 . . .]

MMG for Black scholes'equation with constant volatility [Holtz, Kunoth SINUM 2007]

- B–splines as Finite Element Ansatz functions
- B–spline properties $\rightsquigarrow$
  - Construction of projected Gauss–Seidel smoothing scheme
  - Construction of coarse grid correction

    $\rightsquigarrow$ Optimal complexity of Optimized Coarse Grid Correction Algorithm
- Multigrid convergence and convergence rates
- High accuracy of Greek Letters

Linear Complementary Problem

Variational inequality of elliptic type (in semi–discrete form)

find $u \in \mathcal{K}:\quad a(u, v-u) \geq f(v-u)$ $\qquad$ $a(\cdot,\cdot)$ symmetric, continuous, elliptic

for all $v \in \mathcal{K} := \{v \in H_0^1(\Omega) : \ v(x) \leq g(x)$ for all $x \in \Omega\}$

$\Longleftrightarrow$ linear complementary problem for $u \in S_h \subset H_0^1(\Omega)$

$$\begin{aligned} L_h u_h &\geq f_h && L_h \text{ linear operator} \\ u_h &\leq g_h \in S_h && \text{upper obstacle} \\ (u_h - g_h)(L_h u_h - f_h) &= 0 \end{aligned}$$

solution by Projected Gauss–Seidel Scheme

acceleration by Multigrid Method

Monotone Multigrid Scheme for Linear Complementary Problems

$u_h^{\nu,1} \in S_h$ smooth approximation of exact solution u_h

$$\begin{aligned} L_h u_h &\geq f_h && L_h \text{ linear operator} \\ u_h &\leq g_h \in S_h && \text{upper obstacle} \\ (u_h - g_h)(L_h u_h - f_h) &= 0 \end{aligned}$$

$\Longrightarrow$ Coarse grid problem in S_H $\qquad r,\, \tilde{r} : S_h \to S_H$ restriction operators

$$\begin{aligned} L_H v_H &\geq d_H := r\,(f_h - L_h\, u_h^{\nu,1}) \\ v_H &\leq g_H := \tilde{r}\,(g_h - u_h^{\nu,1}) \\ (v_H - g_H)(L_H v_H - d_H) &= 0 \end{aligned}$$

$\Longrightarrow$ $u_h^{\nu,2} := u_h^{\nu,1} + p\, v_H$ new approximation $\qquad p$ prolongation operator

Requirement: coarse grid correction admissible $\rightsquigarrow$ choose $\tilde{r}$ such that

$$p\, g_H := p\, \tilde{r} \left(g_h - u_h^{\nu,1} \right) \leq g_h - u_h^{\nu,1}$$

$\rightsquigarrow$ **Robust fast solver** $\qquad (\Rightarrow u_h^{\nu,2} := u_h^{\nu,1} + p\, v_H \leq u_h^{\nu,1} + p\, g_H \leq g_h)$

B–Splines

$T := \{\theta_i\}_{i=1,\ldots,n+k}$ (expanded) knot sequence with uniform grid spacing H in interior of $[a,b]$

$$\theta_1 = \ldots = \theta_k = a < \theta_{k+1} < \ldots < \theta_n < b = \theta_{n+1} = \ldots = \theta_{n+k}$$

Recursive definition of B–spline basis function of order k $\quad N_{i,k,T} \qquad i = 1,\ldots,n$

$$N_{i,1,T}(x) = \begin{cases} 1 & \text{if } x \in [\theta_i, \theta_{i+1}) \\ 0 & \text{else} \end{cases} \qquad x \in [a,b]$$

$$N_{i,k,T}(x) = \frac{x - \theta_i}{\theta_{i+k-1} - \theta_i} N_{i,k-1,T}(x) + \frac{\theta_{i+k} - x}{\theta_{i+k} - \theta_{i+1}} N_{i+1,k-1,T}(x) \qquad x \in [a,b]$$

Properties:

- $N_{i,k,T}\big|_{[\theta_i,\theta_{i+1})}$ polynomial of order k
- $\operatorname{supp} N_{i,k,T} \subseteq [\theta_i,\, \theta_{i+k}]$ (local support)
- $N_{i,k,T}(x) \geq 0 \quad$ for all $x \in [a,b]$ (nonnegativity)
- $N_{i,k,T} \in C^{k-2}(I)$ (differentiability)
- $\Sigma := \{N_{1,k,T}, \ldots, N_{n,k,T}\}$ locally independent unconditionally stable basis w.r.t. $\|\cdot\|_{L_p}$, $1 \leq p \leq \infty$, for spline space $\quad S_H :=$ span $\{N_{1,k}, \ldots, N_{n,k}\}$

B–Splines (Contin'd)

Expanding $v \in S_H$: $$v = \sum_{i=1}^{n} v_i \, N_{i,k,T} =: \mathbf{v}^T \mathbf{N}_{k,T}$$

Nonnegativity $\leadsto$

Lemma 1 $v, g \in S_H$ with B–Spline expansion coefficients satisfying $v_i \leq g_i$ for all $i = 1, \ldots, n$

$$\Longrightarrow \quad v(x) \leq g(x) \quad \text{for all } x \in [a, b]$$

Knot sequence $T \leadsto$ finer knot sequence $\Delta := \{\tilde{\theta}_i\}_{i=1,\ldots,\tilde{n}+k}$ with grid spacing $h = \frac{1}{2} H$

$$\tilde{\theta}_1 = \ldots = \tilde{\theta}_k = a < \tilde{\theta}_{k+1} < \ldots < \tilde{\theta}_{\tilde{n}} < b = \tilde{\theta}_{\tilde{n}+1} = \ldots = \tilde{\theta}_{\tilde{n}+k}$$

($\tilde{\theta}_{k+2j} = \theta_{k+j}$ and $\tilde{\theta}_{k+2j-1} = \frac{1}{2}(\theta_{k+j-1} + \theta_{k+j})$ for $j = 1, \ldots, n-k$) $\Longrightarrow$ $\tilde{n} = 2n + 1 - k$

Refinement relation $$N_{i,k,T} = \sum_{j=0}^{k} a_j \, N_{2i-k+j,k,\Delta}$$ with subdivision or mask coefficients

$$a_j := 2^{1-k} \binom{k}{j} \qquad \text{for } j = 0, \ldots, k$$

$$\Longleftrightarrow \quad \mathbf{N}_{k,T} = \mathbf{A}_k^T \, \mathbf{N}_{k,\Delta}$$

$\leadsto$ natural prolongation operator p from $\mathcal{N}_{k,T}$ to $\mathcal{N}_{k,\Delta}$ and restriction operator $r = p^T$

Subdivision matrix

$$\mathbf{A}_k := \begin{pmatrix} a_{k-1} & a_{k-3} & & & & \\ a_k & a_{k-2} & \ddots & & & \\ & a_{k-1} & & a_0 & & \\ & a_k & & a_1 & & \\ & & \ddots & a_2 & & \\ & & & \vdots & & \\ & & & a_{k-1} & \ddots & \\ & & & a_k & & \\ & & & & \ddots & a_0 \\ & & & & & a_1 \end{pmatrix} \in \mathbb{R}^{\tilde{n} \times n}$$

with coefficients $a_j = 2^{1-k} \binom{k}{j}$ for $j = 0, \ldots, k$

$\mathbf{A}_k$ has maximal rank ... follows by properties of B–spline bases

Example: $k = 2$ $k = 3$

$$\mathbf{A}_2 = \begin{pmatrix} 1 & & & & & \\ \frac{1}{2} & \frac{1}{2} & & & & \\ & 1 & & & & \\ & \frac{1}{2} & \frac{1}{2} & & & \\ & & \ddots & \ddots & & \\ & & & \frac{1}{2} & \frac{1}{2} & \\ & & & & 1 & \\ & & & & \frac{1}{2} & \frac{1}{2} \\ & & & & & 1 \end{pmatrix} \in \mathbb{R}^{(2n-1)\times n}$$

$$\mathbf{A}_3 = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & & & & \\ \frac{1}{4} & \frac{3}{4} & & & & \\ & \frac{3}{4} & \frac{1}{4} & & & \\ & \frac{1}{4} & \frac{3}{4} & & & \\ & & \ddots & \ddots & & \\ & & & \frac{3}{4} & \frac{1}{4} & \\ & & & \frac{1}{4} & \frac{3}{4} & \\ & & & & \frac{3}{4} & \frac{1}{4} \\ & & & & \frac{1}{4} & \frac{3}{4} \end{pmatrix} \in \mathbb{R}^{(2n-2)\times n}$$

Solution of linear complementary problem by

Projected Gauss–Seidel Scheme

Given iterate $\mathbf{u}^\nu$ $\rightsquigarrow$ requirement on new iterate: $\mathbf{u}^{\nu+1} := (\mathcal{P} \circ \mathcal{S})\mathbf{u}^\nu \;\leq\; \mathbf{g}$

Realization by $\check{\mathbf{u}}^\nu := \mathcal{S}(\mathbf{u}^\nu)$ $\mathcal{S}$ Gauss–Seidel iteration

and projection step $\mathbf{u}^{\nu+1} := \mathcal{P}(\check{\mathbf{u}}^\nu)$ where

$\mathcal{P}u_i := \min\{u_i, g_i\}$ $\quad i = 1, \ldots, n$ B-spline expansion coefficients

$k = 2$: projection step is $\mathcal{P}u(\tilde{\theta}_i) := \min\{u(\tilde{\theta}_i), g(\tilde{\theta}_i)\}$ on point values

Coarse Grid Approximations of Obstacles

Given: cardinal B–Spline $\tilde{S}$ of order k with $\tilde{n}$ d.o.f. on uniform fine grid $\Delta \subset \Omega := (0,1)$

Goal: construct monotone coarse grid approximation S with n d.o.f. on coarse grid $T \subset \Delta$

Requirements:

(1) $S(x) \leq \tilde{S}(x)$ for all $x \in \Omega$ (monotonicity)

(2) $S(x) \geq L_k(x)$ for all $x \in \Omega$ for some lower bound L_k (quasi–optimality)

(3) $S \approx \tilde{S}$

(4) $O(n)$ arithmetic operations (optimal complexity of coarse grid)

Representation of Obstacles in B–Spline Bases

$$\tilde{S}(x) = \sum_{i=1}^{\tilde{n}} \tilde{c}_i \, N_{i,k,\Delta}(x) \qquad\qquad S(x) = \sum_{i=1}^{n} c_i \, N_{i,k,T}(x)$$

Proposition 1 (Monotone coarse grid approximation)

$$\mathbf{A}_k \, \mathbf{c} \leq \tilde{\mathbf{c}} \quad \Longrightarrow \quad S(x) \leq \tilde{S}(x) \qquad \text{for all } x \in \Omega$$

$\mathbf{A}_k \in \mathbb{R}^{\tilde{n} \times n}$ refinement matrix with entries $a_j = 2^{1-k} \binom{k}{j}$ $\quad j = 0, \ldots, k$

Rows of $\mathbf{A}_k$ sum to 1 & Proposition 1 $\Longrightarrow$

Proposition 2 $L_k := \mathbf{q}^T \mathbf{N}_{k,T}$
with expansion coefficients $q_i := \min\{\tilde{c}_{2i-k}, \ldots, \tilde{c}_{2i}\} \quad i = 1, \ldots, n$
is monotone coarse grid correction for $\tilde{S}$

Remark: $k = 2$: approximation from [Mandel 1984]

Definition: Quasi–optimal: any coarse grid correction 'better' than L_k

(Motivation: MMG methods with quasi–optimal obstacles reduce asumptotically to linear relaxation)

Find 'better' approximations than $L_k \rightsquigarrow$

Linear Optimization Problem (LOP): given $\mathbf{q} \in \mathbb{R}^n \quad \mathbf{A}_k \in \mathbb{R}^{\tilde{n} \times n} \quad \tilde{\mathbf{c}} \geq 0 \quad \tilde{n} > n$

$$\begin{aligned} \text{minimize} \quad & F(\mathbf{c}) := \sum_{\theta \in T} |\tilde{S}(\theta) - S(\theta)| = \sum_{\theta \in T} \left| \tilde{\mathbf{c}}^T \mathbf{N}_{k,\Delta}(\theta) - \mathbf{c}^T \mathbf{N}_{k,T}(\theta) \right| \\ \text{subject to} \quad & \mathbf{A}_k \, \mathbf{c} \leq \tilde{\mathbf{c}} \qquad \text{(monotonicity)} \\ & \mathbf{c} \geq \mathbf{q} \qquad \text{(quasi–optimality)} \end{aligned}$$

Theorem (Optimized Coarse Grid Correction (OCGC))

Define $$\hat{b}_{j,i} := \tilde{c}_j - \sum_{\ell=1}^{i-1} a_{j\ell}\,\tilde{c}_\ell - \sum_{\ell=i+1}^{\lceil \frac{j}{2} \rceil} a_{j\ell}\,q_\ell \qquad \text{for } i = 1, \ldots, n,\ j > i$$

and $$\tilde{b}_i := \tilde{c}_i - \sum_{j=1}^{\lfloor \frac{i-1}{2} \rfloor} a_{ij} c_j$$

$\Longrightarrow$

(I) recursively defined coefficients

$$c_i := \min \left\{ \frac{\tilde{b}_{2i-1}}{a_{2i-1,\,i}}, \frac{\tilde{b}_{2i}}{a_{2i,\,i}}, \frac{\hat{b}_{2i+1,i}}{a_{2i+1,\,i}}, \frac{\hat{b}_{2i+2,i}}{a_{2i+2,\,i}}, \ldots, \frac{\hat{b}_{m,i}}{a_{m,\,i}} \right\} \qquad \text{for } i = 1, \ldots, n$$

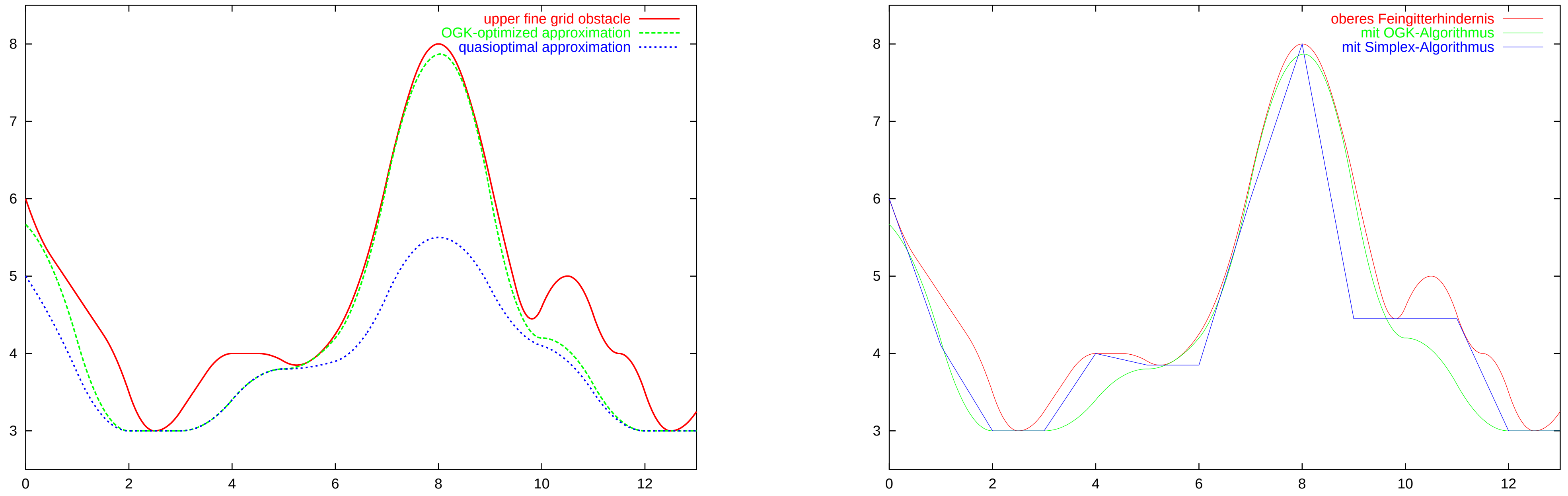
solve (LOP)

(II) OCGC Algorithm needs $O(n)$ operations (optimal complexity)

$k = 2$: direct solution by Fourier–Motzkin elimination; recovers Kornhuber's geometric construction

$k > 2$: exploits properties of $\mathbf{A}_k$

Results of OCGC Approximation



OCGC approximation for $k = 3$ and comparison with simplex method (not $\mathcal{O}(n)$)

Monotone Multigrid Scheme

MMG$_\ell$ (ν– th cycle on level $\ell \geq 1$)

$u_\ell^\nu \in S_\ell$ given approximation

1. Projected Gauss–Seidel scheme (PSOR): $u_\ell^{\nu,1} := (\mathcal{P} \circ \mathcal{S}(u_\ell^\nu))^{\eta_1}$
2. Coarse grid correction:

$$d_\ell := f_\ell - \mathcal{L}_\ell u_\ell^{\nu,1}$$

$$f_{\ell-1} := r d_\ell$$

$$g_{\ell-1} := \tilde{r}(g_\ell - u_\ell^{\nu,1})$$

$$\mathcal{L}_{\ell-1} := r \mathcal{L}_\ell p$$

If $\ell = 1$, solve linear complementary problem

$$\begin{array}{rcl} \mathcal{L}_{\ell-1} v & \geq & f_{\ell-1} \\ v & \leq & g_{\ell-1} \\ (v - g_{\ell-1})(\mathcal{L}_{\ell-1} v - f_{\ell-1}) & = & 0 \end{array}$$

exactly; set $v_{\ell-1} := v$

If $\ell > 1$, do γ steps of **MMG**$_{\ell-1}$ with initial value $u_{\ell-1}^0 := 0$ and solution $v_{\ell-1}$

Set $u_\ell^{\nu,2} := u_\ell^{\nu,1} + p v_{\ell-1}$

3. Projected Gauss–Seidel scheme (PSOR): $u_\ell^{\nu,3} := (\mathcal{P} \circ \mathcal{S}(u_\ell^{\nu,2}))^{\eta_2}$

Set $u_\ell^{\nu+1} := u_\ell^{\nu,3}$

Convergence Theory for Monotone Multigrid Method

Kornhuber 1994, 1997: Interpretation as subspace correction method

(assuming monotone and quasioptimal approximation to obstacles)

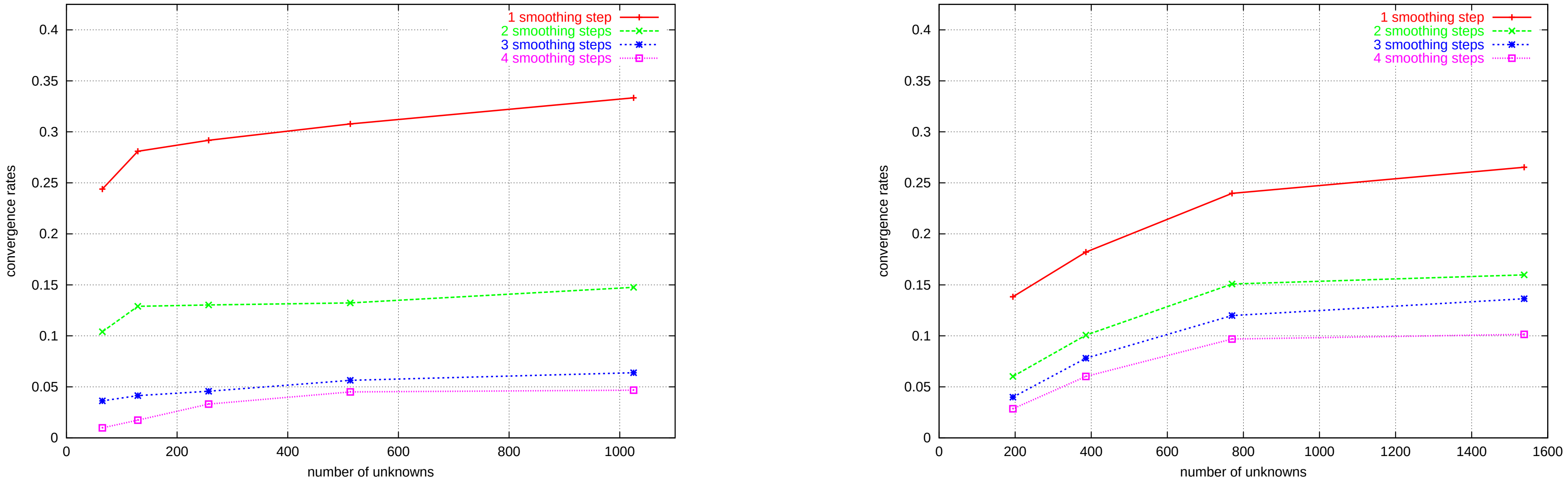
— carries over to higher order B–spline case

(replacing function values for $k = 2$ by B–spline expansion coefficients)

- Global convergence of monotone multigrid method
- Identification of contact set after finite number of iterations
- After identification of contact set: reduction of method to linear relaxation – interpretation as linear multiplicative Schwarz scheme
- Approximate extended relaxation with search directions from coarser grid; truncated version: coarse grid functions adapted to contact set

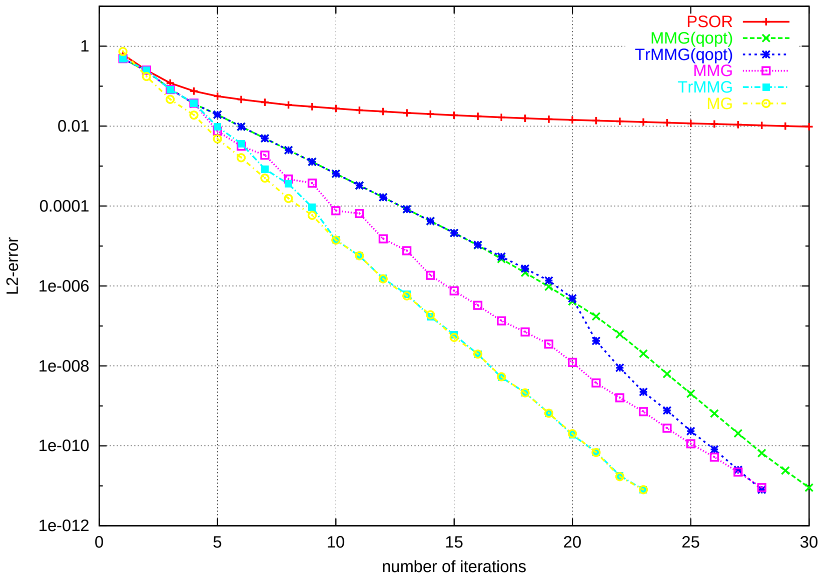
Extension to $d > 1$ by tensor products

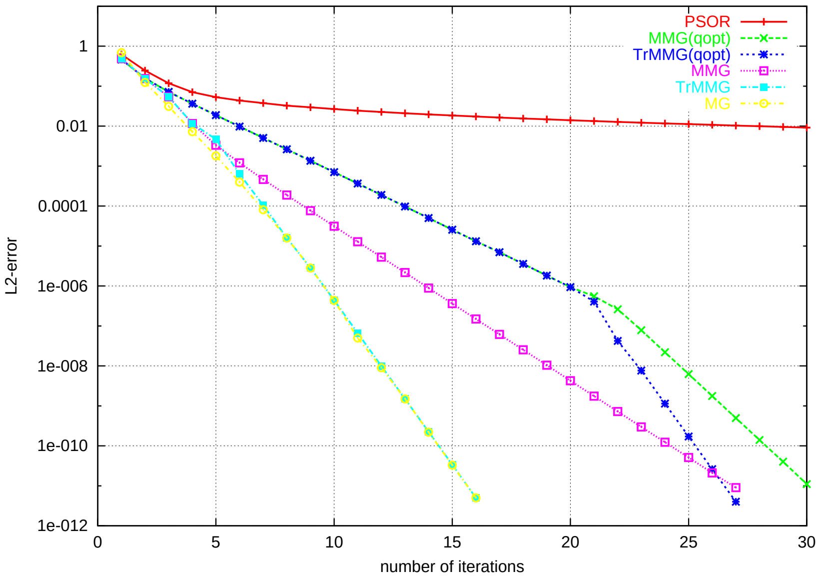
Monotone Multigrid Method: Convergence Rates



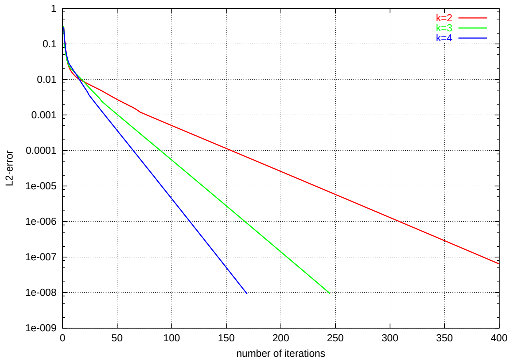
Convergence rates for monotone multigrid method for $k = 2$ and $k = 3$

Convergence Behaviour



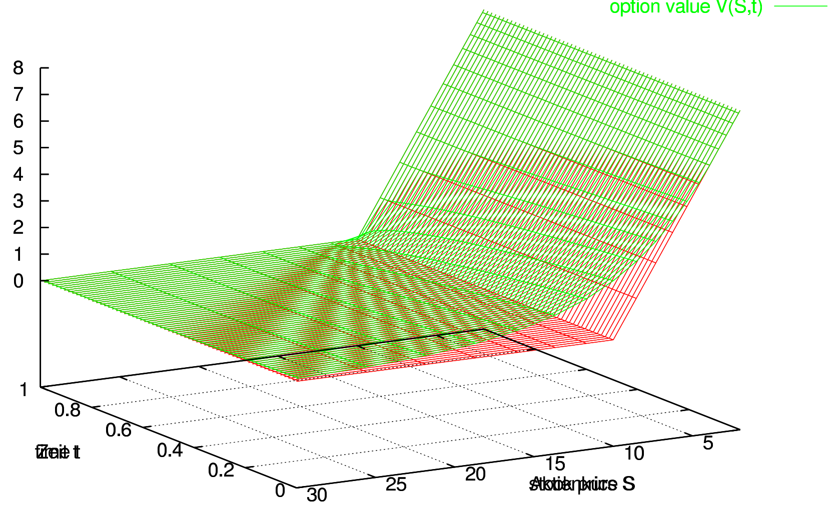


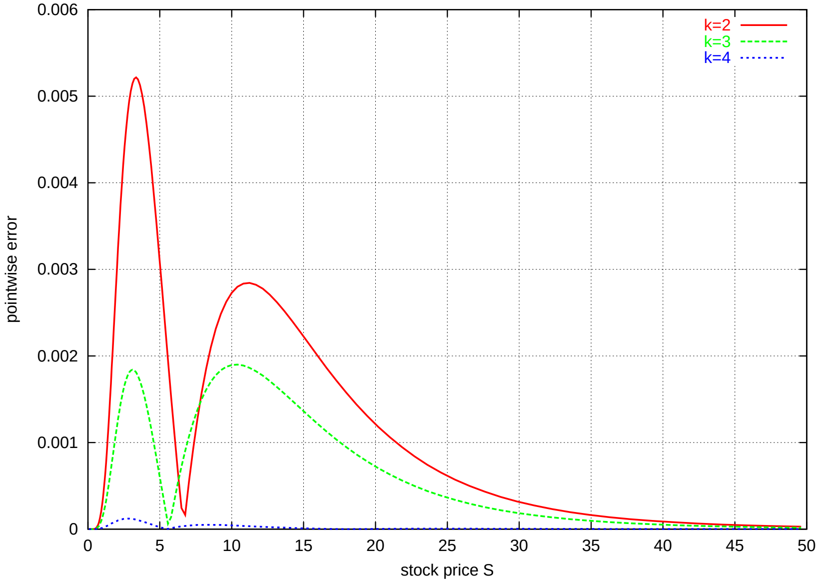
Comparison of different iterative schemes for $k = 2$ and $k = 3$



Comparison of PSOR iteration errors for one time step for $k \in \{2, 3, 4\}$

Option Pricing Computations and Approximation of Greek Letters (with Constant Volatility)





Prize $V(S,t)$ of American Put Option for constant volatility $\sigma = 0.6$

Pointwise error of Gamma $= \frac{\partial^2 V(S,0)}{\partial S^2}$ for American option prizing problem

Convergence Behaviour and Prize of American Put Option (with Stochastic Volatility)