Strong and weak error estimates for the solution of an elliptic partial differential equation with random coefficients

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Equation

- D a bounded $\mathcal{C}^2$ domain of $\mathbb{R}^d$, $(\Omega, \mathcal{F}, P)$ a probability space
- $a : \Omega \times D \to \mathbb{R}$ a lognormal homogeneous random field $a(\omega, x) = e^{g(\omega,x)}$ where g is a gaussian homogeneous mean-free random field with $cov[g](x, y) = k(\|x - y\|)$, $k \in \mathcal{C}^{0,1}(\mathbb{R})$
- We look for $u : \Omega \times D \to \mathbb{R}$ such that for almost every ω

$$\begin{aligned} -\nabla.(a(\omega,.)\nabla u(\omega,.)) &= f(x) \text{ on } D \\ u(\omega,.) &= 0 \text{ on } \partial D. \end{aligned} \quad (1)$$

Proposition

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$a_{min}(\omega) = \min_{x \in D} a(\omega, x)$ *and* $a_{max}(\omega) = \max_{x \in D} a(\omega, x)$.

Then $\frac{1}{a_{min}(\omega)} \in L^p(\Omega)$ *and* $a_{max}(\omega) \in L^p(\Omega)$ $\forall p > 0$.

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Proposition

The equation 1 admits a unique solution $u \in L^p(\Omega, H_0^1(D))$, $\forall p > 0$.

Approximation of a

We denote by g_N the troncated Karhunen-Loève expansion of g troncated at order N, a_N its exponential:

$$
\begin{aligned}
g_N(\omega, x) &= \sum_{n=1}^{N} \sqrt{\lambda_n} b_n(x) Y_n(\omega) \\
a_N(\omega, x) &= e^{\sum_{n=1}^{N} \sqrt{\lambda_n} b_n(x) Y_n(\omega)}.
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Remark: The $(Y_n)_{n \geq 1}$ are independent.

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Assumptions:

- the eigenfunctions b_n are continuously differentiable with $\|b_n\|_\infty \leq C$ and $\|b_n'\|_\infty \leq Cn^a$
- $\sum_{n\geq 1} \lambda_n n^b < +\infty$ for some $b > 0$.

Strong convergence of a_N to a

- $\forall p > 0$, $\forall 0 < \alpha < \min\{b, 2a\}$

$$||g_N - g||_{L^p(\Omega, \mathcal{C}^0(D))} \leq A_{\alpha,p} \sqrt{\sum_{n>N} \lambda_n n^\alpha} \quad \forall N \in \mathbb{N}.$$

Strong convergence of a_N to a

- $\forall p > 0$, $\forall 0 < \alpha < \min\{b, 2a\}$

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- For almost all ω, $g_N \underset{\mathcal{C}^0(D)}{\longrightarrow} g$ and so $a_N \underset{\mathcal{C}^0(D)}{\longrightarrow} a$ as $N \to +\infty$.

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- We define $a_N^{min}(\omega) = \min_{x \in D} a_N(\omega, x)$ and $a_N^{max}(\omega) = \max_{x \in D} a_N(\omega, x)$ a.s. Then for all $p > 0$,

$$\left\| \frac{1}{a_N^{min}} \right\|_{L^p(\Omega)} \leq B_p \text{ and } \|a_N^{max}\|_{L^p(\Omega)} \leq B_p \quad \forall N \in \mathbb{N}.$$

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- $\forall p > 0$, $\forall 0 < \alpha < \min\{b, 2a\}$

$$\|a_N - a\|_{L^p(\Omega,\mathcal{C}^0(D))} \leq C_{\alpha,p}\sqrt{\sum_{n>N} \lambda_n n^\alpha} \quad \forall N \in \mathbb{N}.$$

Strong convergence of u_N to u

We define the approximation u_N of u as the solution of:

$$\begin{aligned} -\nabla.(a_N(\omega,.)\nabla u_N(\omega,.)) &= f(x) \text{ on } D \\ u_N(\omega,.) &= 0 \text{ on } \partial D. \end{aligned}$$

Then : $u_N(\omega, x) = u_N(Y_1(\omega), ..., Y_N(\omega), x)$ (Doob-Dynkin lemma).

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Proposition

$\forall p > 0$, $\forall 0 < \alpha < \min\{b, 2a\}$

$$\|u_N - u\|_{L^p(\Omega,H_0^1(D))} \leq F_{\alpha,p}\sqrt{\sum_{n>N}\lambda_n n^\alpha} \quad \forall N \in \mathbb{N}.$$

Weak convergence of u_N to u

Proposition

There exists a constant C such that for any $\varphi \in \mathcal{C}^4(\mathbb{R}, \mathbb{R})$ whose derivatives are bounded by a constant C_φ

$$||\mathbb{E}_\omega[\varphi(u) - \varphi(u_N)]||_{H_0^1} \leq CC_\varphi \sum_{n>N} \lambda_n.$$

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Remark: The weak order is twice the strong order.

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Sketch of the proof: For any multi-index $\alpha \in \mathbb{N}^N$ with finite support

$$\left\| \frac{\partial^\alpha u_N(y, x)}{\partial y^\alpha} \right\|_{H_0^1(D)} \leq k_{|\alpha|} \sqrt{\frac{a_{max}^N(y)}{a_{min}^N(y)}} ||u_N||_{H_0^1} C^{|\alpha|} \prod_{i \in \mathbb{N}} \sqrt{\lambda_i^{\alpha_i}}.$$

$$u(\omega, x) - u_N(\omega, x)$$
$$= u(Y_1(\omega), ..., Y_N(\omega), Y_{N+1}(\omega), ..., x) - u(Y_1(\omega), ..., Y_N(\omega), 0, ..., x)$$

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&= \sum_{i>N} \frac{\partial u}{\partial y_i}(Y_1(\omega), ..., Y_N(\omega), 0, ..., x) Y_i(\omega) \\
&+ \sum_{i \neq j > N} \frac{\partial^2 u}{\partial y_i \partial y_j}(Y_1(\omega), ..., Y_N(\omega), 0, ..., x) Y_i(\omega) Y_j(\omega)
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$$+ \sum_{i>N} \frac{\partial^2 u}{\partial y_i^2}(Y_1(\omega), ..., Y_N(\omega), 0, ..., x) Y_i(\omega)^2 + ...$$

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\end{aligned}
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The independence of the Y_i yields:

$$
\mathbb{E}[u - u_N](x) = 0 + \sum_{i>N} \mathbb{E}\left[\frac{\partial^2 u}{\partial y_i^2}(Y_1(\omega), ..., Y_N(\omega), 0, ..., x)\right] + ...
$$

Example: the 1D exponential covariance kernel case

We take $D = (0,1)$ and $cov[g](x,y) = \sigma^2 e^{-\frac{|x-y|}{l}}$ where l is the correlation length. Then we have analytic expressions for the eigenvalues λ_n and the eigenfunctions b_n, in particular :

- $\lambda_n \underset{n \to +\infty}{\sim} \frac{2\sigma^2}{l\pi^2 n^2}$
- $\forall n \in \mathbb{N}$, $\|b_n\|_\infty \leq C$ and $\|b_n'\|_\infty \leq Cn$.

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Proposition (Strong convergence result)

$\forall p > 0$, $\forall 0 < \alpha < 1$

$$\|u_N - u\|_{L^p(\Omega, H_0^1(D))} \leq F_{\alpha,p} N^{\frac{\alpha-1}{2}} \quad \forall N \in \mathbb{N}.$$

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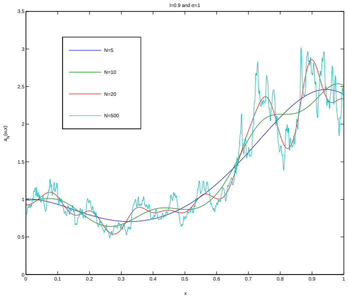
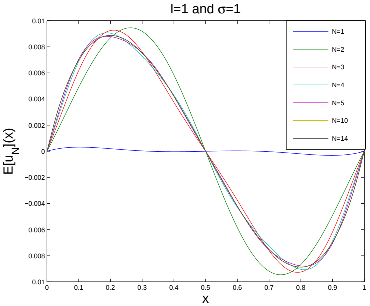
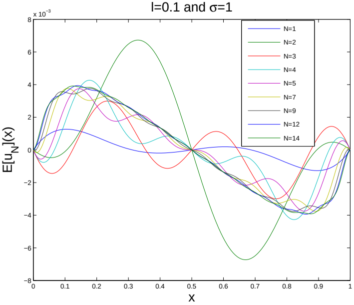
$$||\mathbb{E}_\omega[\varphi(u) - \varphi(u_N)]||_{H_0^1(D)} \leq C_\varphi \frac{C}{N}.$$

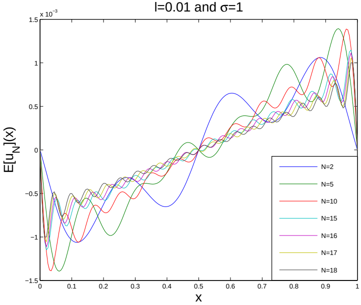




$a_N(\omega, x)$ for different values of N

$\mathbb{E}[u_N(x)]$ for different values of N
here we have $\|E[u - u_N]\|_{L^2(D)} \simeq \frac{c}{N^{2.7}}$.





$\mathbb{E}[u_N(x)]$ for different values of N for $l = 0.1$

$\mathbb{E}[u_N(x)]$ for different values of N for $l = 0.01$

Example: the analytic covariance kernel case

We suppose that $cov[g]$ is analytic on D^2, then we have

Theorem (Frauenfelder, Schwab, Todor)

- $$\lambda_n \leq c_1 e^{-c_2 n^{1/d}} \quad \forall n \in \mathbb{N}$$
- *for any $s > 0$ there exists a constant c_s such that,*
$$\|b_n\|_\infty \leq c_s |\lambda_n|^{-s} \text{ and } \|b'_n\|_\infty \leq c_s |\lambda_n|^{-s} \quad \forall n \in \mathbb{N}.$$

We have then strong and weak convergence results, analogous to the previous results.

Proposition (Strong convergence result)

For any $0 < s < \frac{1}{2}$*, and* $p > 0$

$$\|u - u_N\|_{L^p(\Omega, H_0^1(D))} \leq H_{s,p} \sqrt{\sum_{n>N} \lambda_n^{1-2s}} \quad \forall N \in \mathbb{N}$$

therefore

$$\|u - u_N\|_{L^p(\Omega, H_0^1(D))} \leq I_{d,s,p} N^{\frac{d-1}{2d}} e^{-\frac{c_2(1-2s)}{2} N^{1/d}} \quad \forall N \in \mathbb{N}$$

Proposition (Weak convergence result)

For any $0 < s < \frac{1}{2}$, *for all* $\varphi \in \mathcal{C}^4(\mathbb{R},\mathbb{R})$ *whose derivatives are bounded by a constant* C_φ, *we have:*

$$\|\mathbb{E}[\varphi(u_N) - \varphi(u)]\|_{H_0^1(D)} \leq J_s C_\varphi \sum_{n>N} \lambda_n^{1-2s} \quad \forall N \in \mathbb{N}$$

therefore

$$\|\mathbb{E}[\varphi(u_N) - \varphi(u)]\|_{H_0^1(D)} \leq K_{d,s} C_\varphi N^{\frac{d-1}{d}} e^{-c_2(1-2s)N^{1/d}} \quad \forall N \in \mathbb{N}.$$