On The Hausdorff Dimension of Bernoulli Convolutions.

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June 2018

Beta Expansions

- Given $\beta > 1$, define $\pi_\beta : \{0,1\}^{\mathbb{N}} \to I_\beta = \left[0, \frac{1}{\beta-1}\right]$ by

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- For ν_β a.e. x, $\dim_H(\nu_\beta) = \lim_{r \to 0} \frac{\log(\nu_\beta(B(x,r)))}{\log r}$.

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- T preserves measure ν_β. Can study both X_β and ν_β using T, e.g. Dimension formulae, entropy, Lyapunov exponents...

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- We could try to define T the same way:

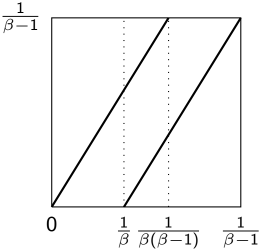

Figure : $T(\sum_{i=1}^{\infty} a_i \beta^{-i}) = \sum_{i=1}^{\infty} a_{i+1} \beta^{-i}$ is ambiguous

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This gives some imperfect upper bounds in multifractal analysis of ν_β.

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We can use this self similarity equation to define a Lebesgue measure preserving system.

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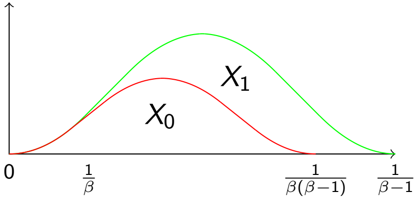

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Then define the function $\phi : X \to X$ by

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Theorem

The set $\pi_\beta^{-1}(x)$ has positive finite $\left(\frac{\frac{\log 2}{\log \beta}}{\log 2}\right)$-dimensional Hausdorff measure for almost every $x \in I_\beta$ if and only if ν_β is absolutely continuous with bounded density.

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- The Garsia entropy $H(\beta) := \lim_{n\to\infty} \frac{1}{n} H_n(\beta)$.

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- Salem if all other Galois conjugates have $\beta_i \leq 1$ with at least one of absolute value 1.

- (Hochman '14) $\dim_H(\nu_\beta) < 1$ implies either that β is algebraic or that min distance between different elements of X_n tends to 0 superexponentially:

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- If Pisot numbers are the only **algebraic** $\beta \in (1,2)$ with $\dim_H(\nu_\beta) < 1$, then they are the only $\beta \in (1,2)$ with $\dim_H(\nu_\beta) < 1$.

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Theorem (Akiyama, Feng, K., Persson)

If $\beta \in (1,2)$ is algebraic with l conjugates of absolute value 1, there exists a constant C such that

$$\frac{1}{n} H_n(\beta) - \frac{C + l \log(n+1)}{n} \leq H(\beta) \leq \frac{1}{n} H_n(\beta).$$

Corollary

If $\beta \in (1,2)$ *is a hyperbolic, non-Pisot solution of a* $0, \pm 1$ *polynomial of degree* ≤ 5 , $\dim_H(\nu_\beta) = 1$.

(27 such examples)

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Very rapid estimation of dimension possible. For the (Pisot) β satisfying

$$\beta^4 - \beta^3 - 1 = 0$$

$\dim_H(\nu_\beta) \in (0.99999, 1)$.

Understanding Coincidences

$$
\begin{aligned}
\sum_{i=1}^{n} a_i \beta^{-i} &= \sum_{i=1}^{n} b_i \beta^{-i} \\
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Understanding Coincidences

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$a_i, b_i \in \{0,1\}$, $\epsilon_i := a_i - b_i \in \{-1,0,1\}$.
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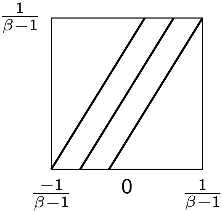

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$$T_{\epsilon_m} \circ T_{\epsilon_{n-1}} \circ \cdots \circ T_{\epsilon_1}(0) \in \left[\frac{-1}{\beta - 1}, \frac{1}{\beta - 1} \right].$$

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Fact 2: If β is Pisot then the set

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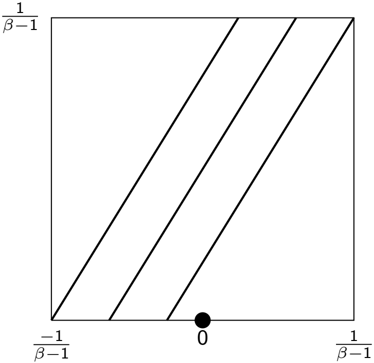
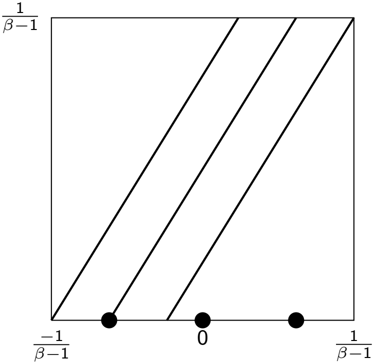
So we have a finite graph $\mathcal{G}$ with vertex set $V(\beta)$, edges labelled by $\{-1, 0, 1\}$ such that $\sum_{i=1}^{n} a_i \beta^{-i} = \sum_{i=1}^{n} b_i \beta^{-i}$ if and only if there is a path from 0 to 0 in $\mathcal{G}$ made by following edges labelled $a_i - b_i$.


$\frac{1}{\beta-1}$
$\frac{-1}{\beta-1}$
0
$\frac{1}{\beta-1}$



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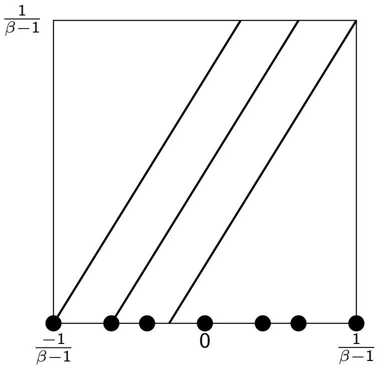
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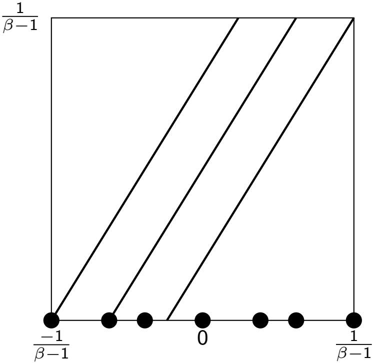
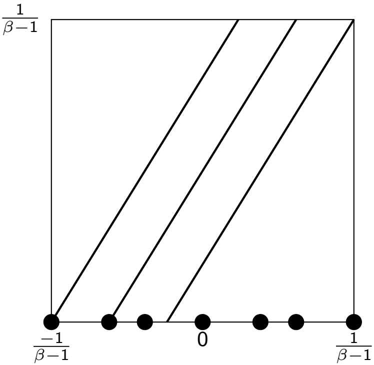


We can count all pairs $a_1 \cdots a_n, b_1 \cdots b_n$ with

$$\sum_{i=1}^{n} a_i \beta^{-i} = \sum_{i=1}^{n} b_i \beta^{-i}$$

by counting paths from 0 to 0 under these dynamics.



We want to fix $\underline{a}$ and count all $b_1 \cdots b_n$ with

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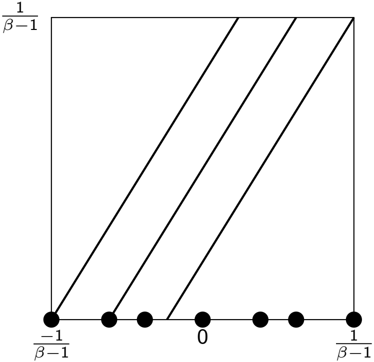

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To do this make two subgraphs. $a_i = 1 \implies a_i - b_i \in \{0, 1\}$.
$a_i = 0 \implies a_i - b_i \in \{-1, 0\}$.

Two Matrices

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$\frac{1}{n} H_n(\beta)$ decreases to limit $H(\beta)$.

Lower Bounds

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This gives us the rate at which $\frac{1}{n}H_n(\beta)$ converges to $H(\beta)$.

The Hyperbolic Non-Pisot Case

Fact 3: Let β be algebraic and let β_2 be conjugate to β. Then

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So we can also look at the dynamics using β_2, and disregard orbits for which

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Fact 4: If $\beta \in (1,2)$ is hyperbolic, with conjugates $\beta_2, \cdots \beta_k$ larger than one in modulus, then the set $V(\beta)$ of forward orbits

$$T_{\epsilon_n} \circ \cdots T_{\epsilon_1}(0)$$

with the restriction that for each $j \in \{1, \cdots, k\}$,

$$\sum_{i=1}^{n} \epsilon_i \beta_j^{n-i} \leq |\frac{1}{\beta_j - 1}|.$$

is finite. So everything works as in the Pisot case.

The Non-Hyperbolic Case

The set $V(\beta)$ and the resulting matrices our now infinite. But we still get a rate of convergence if we mess around a bit.