Subadditive ergodic theory and applications

Antti Käenmäki

Edinburgh, 18th June 2018

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This is an introductory talk to the topic
(from one perspective)

Birkhoff ergodic theorem

Theorem (Birkhoff, 1931)

Let $T\colon X \to X$ and μ be an ergodic T-invariant probability measure. If $f \in L^1(\mu)$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = \int f \, \mathrm{d}\mu$$

for μ-almost all $x \in X$.

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for μ-almost all $x \in X$.

- The result can be generalized by recalling the Fekete's lemma for subadditive sequences $(S_n)_{n\in\mathbb{N}}$: If

$$S_{m+n} \le S_m + S_n$$

for all $m, n \in \mathbb{N}$, then

$$\lim_{n\to\infty} \tfrac{1}{n} S_n = \inf_{n\in\mathbb{N}} \tfrac{1}{n} S_n.$$

Kingman's subadditive ergodic theorem

Theorem (Kingman, 1968)

Let $T\colon X \to X$ and μ be an ergodic T-invariant probability measure. If $(S_n)_{n\in\mathbb{N}}$ is a subadditive sequence of $L^1(\mu)$ functions, i.e.

$$S_{m+n} \le S_m + S_n \circ T^m$$

for all $m, n \in \mathbb{N}$, then

$$\lim_{n\to\infty} \frac{1}{n} S_n(x) = \lim_{n\to\infty} \frac{1}{n} \int S_n \,\mathrm{d}\mu = \inf_{n\in\mathbb{N}} \frac{1}{n} \int S_n \,\mathrm{d}\mu$$

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for μ-almost all $x \in X$.

- If $f \in L^1(\mu)$, then choosing $S_n = \sum_{k=0}^{n-1} f \circ T^k$ above gives the Birkhoff ergodic theorem.

Subadditive potential

- Consider the semigroup generated by the matrix tuple $\mathsf{A} = (A_1, \ldots, A_N) \in GL_d(\mathbb{R})^N$.

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- Let $\Sigma = \{1, \ldots, N\}^{\mathbb{N}}$ and if $\mathtt{i} = i_1 i_2 \cdots \in \Sigma$, then write $\mathtt{i}|_n = i_1 \cdots i_n$ and $\Sigma_n = \{\mathtt{i}|_n : \mathtt{i} \in \Sigma\}$ for all $n \in \mathbb{N}$.

  If $\mathtt{i} \in \Sigma_n$ for some n, then we write $[\mathtt{i}] = \{\mathtt{j} \in \Sigma : \mathtt{j}|_n = \mathtt{i}\}$.

  We use $\Sigma_* = \bigcup_{n \in \mathbb{N}} \Sigma_n$ to index the elements in the semigroup, that is, $A_{\mathtt{i}} = A_{i_1} \cdots A_{i_n}$ for all $\mathtt{i} = i_1 \cdots i_n \in \Sigma_n$ and $n \in \mathbb{N}$.

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- Let $\sigma \colon \Sigma \to \Sigma$, $\sigma(i_1 i_2 i_3 \cdots) = i_2 i_3 \cdots$, be the left-shift, $s > 0$, and consider the functions

  $$\mathtt{i} \mapsto \log \|A_{\mathtt{i}|_n}\|^s$$

  for all $n \in \mathbb{N}$. The sequence $\Phi^s = (\log \|A_{\mathtt{i}|_n}\|^s)_{n \in \mathbb{N}}$ is subadditive.

Lyapunov exponent

- If μ is a σ-invariant probability measure on Σ, then the number

$$\lambda(\Phi^s, \mu) = \inf_{n \in \mathbb{N}} \frac{1}{n} \sum_{\mathtt{i} \in \Sigma_n} \mu([\mathtt{i}]) \log \|A_{\mathtt{i}}\|^s = \lim_{n \to \infty} \frac{1}{n} \sum_{\mathtt{i} \in \Sigma_n} \mu([\mathtt{i}]) \log \|A_{\mathtt{i}}\|^s$$

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- If μ is ergodic and σ-invariant, then Kingman's subadditive ergodic theorem implies

$$\lim_{n \to \infty} \frac{1}{n} \log \|A_{\mathtt{i}|_n}\| = \lambda(\Phi^1, \mu)$$

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  for μ-almost all $\mathtt{i} \in \Sigma$.

- The existence of the Lyapunov exponent was proved by Furstenberg & Kesten (1960). The number describes the growth rate of $\|A_{\mathtt{i}|_n}\|$ for a μ-generic $\mathtt{i} \in \Sigma$.

Thermodynamic formalism

- If $\Psi = (S_n)_{n\in\mathbb{N}}$ is a subadditive sequence of $L^1(\mu)$ functions, then the *pressure* of Ψ is defined by

$$P(\Psi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\mathtt{i}\in\Sigma_n} \exp \max_{\mathtt{j}\in[\mathtt{i}]} S_n(\mathtt{j}).$$

Recall that the *Kolmogorov-Sinai entropy* of μ is

$$h(\mu) = -\lim_{n\to\infty} \frac{1}{n} \sum_{\mathtt{i}\in\Sigma_n} \mu([\mathtt{i}]) \log \mu([\mathtt{i}]).$$

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$$h(\mu) = -\lim_{n\to\infty} \frac{1}{n} \sum_{\mathtt{i}\in\Sigma_n} \mu([\mathtt{i}]) \log \mu([\mathtt{i}]).$$
- It is easy to see that $P(\Phi^s) \geq h(\mu) + \lambda(\Phi^s, \mu)$ for all σ-invariant probability measures μ.

Theorem (K., 2004)

There exists an ergodic σ-invariant probability measure μ such that $P(\Phi^s) = h(\mu) + \lambda(\Phi^s, \mu)$. Any σ-invariant measure satisfying this equality is called an *equilibrium state* for Φ^s.

Equilibrium states

- It is remarkable to notice that there can be two distinct ergodic equilibrium states for Φ^s. For example, let

$$A_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{pmatrix},$$

  and choose $s > 0$ such that $2^{-s} + 4^{-s} = 1$. Then the Bernoulli measures obtained from the probability vectors $(2^{-s}, 4^{-s})$ and $(4^{-s}, 2^{-s})$ are ergodic equilibrium states for Φ^s.

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Theorem (Feng & K., 2011)

If $s > 0$ and $\mathsf{A} \in GL_d(\mathbb{R})^N$, then the maximum possible number of distinct ergodic equilibrium states for Φ^s is d and every equilibrium state is fully supported.

Equilibrium states

- A tuple $\mathsf{A} = (A_1, \ldots, A_N) \in GL_d(\mathbb{R})^N$ is *irreducible* if there is no non-trivial proper linear subspace V of $\mathbb{R}^d$ such that $A_i V = V$ for all $i \in \{1, \ldots, N\}$; otherwise A is *reducible*.

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- A tuple $\mathsf{A} \in GL_d(\mathbb{R})^d$ is *strongly irreducible* if V above can be replaced by a finite union of non-trivial proper linear subspaces.
- The idea of the proof of the previous theorem is to show that the tuple A is a tuple of block-upper diagonal matrices in some basis, and then observe that each block in the diagonal is irreducible and contributes one ergodic equilibrium state.

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- The idea of the proof of the previous theorem is to show that the tuple A is a tuple of block-upper diagonal matrices in some basis, and then observe that each block in the diagonal is irreducible and contributes one ergodic equilibrium state.

- To study the properties of equilibrium states, we may thus assume that the tuple A is irreducible.

Characterization of equilibrium states

- We say that a probability measure μ on Σ is a *Gibbs-type measure* for Φ^s if there exists a constant $C \geq 1$ such that

$$C^{-1} \leq \frac{\mu([\mathtt{i}])}{\exp\bigl(-nP(\Phi^s)) + \log \|A_{\mathtt{i}}\|^s\bigr)} \leq C$$

  for all $\mathtt{i} \in \Sigma_n$ and $n \in \mathbb{N}$.

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Theorem (Feng & K., 2011)

If $s > 0$ and $\mathsf{A} \in GL_d(\mathbb{R})^N$ is irreducible, then there is only one equilibrium state for Φ^s and it is a Gibbs-type measure for Φ^s.

Characterization of equilibrium states

- Recall that $A \in GL_2(\mathbb{R})$ is *elliptic* if there are an invertible conjugation matrix M and $c \neq 0$ such that $cMAM^{-1}$ is an orthogonal matrix. We say that $\mathsf{A} \in GL_2(\mathbb{R})^N$ is *strongly elliptic* if all the elements of A are elliptic with respect to the same conjugation matrix.

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- Let $\mathbb{RP}^1$ denote the real projective line, which is the set of all lines through the origin in $\mathbb{R}^2$. We call a proper subset $\mathcal{C} \subset \mathbb{RP}^1$ a *multicone* if it is a finite union of closed projective intervals. We say that $\mathsf{A} \subset GL_2(\mathbb{R})$ has a *strongly invariant multicone* if there exists a multicone $\mathcal{C} \subset \mathbb{RP}^1$ such that $A\mathcal{C} \subset \mathcal{C}^o$ for all $A \in \mathsf{A}$.

Characterization of equilibrium states

- A probability measure μ on Σ is *quasi-Bernoulli* if there exists a constant $C \geq 1$ such that

$$C^{-1}\mu([\mathtt{i}])\mu([\mathtt{j}]) \leq \mu([\mathtt{i}\,\mathtt{j}]) \leq C\mu([\mathtt{i}])\mu([\mathtt{j}])$$

for all $\mathtt{i}, \mathtt{j} \in \Sigma_*$.

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Theorem (Bárány & Morris & K., preprint)

If $\mathsf{A} \in GL_2(\mathbb{R})^N$ is irreducible and μ is the equilibrium state for Φ^s, then the following are equivalent:

(1) μ is a quasi-Bernoulli measure,
(2) $\|A_{\mathtt{i}}A_{\mathtt{j}}\| \geq c\|A_{\mathtt{i}}\|\|A_{\mathtt{j}}\|$ for all $\mathtt{i}, \mathtt{j} \in \Sigma_*$,
(3) $\mathsf{A} = \mathsf{A}_e \cup \mathsf{A}_h$, where A_e is strongly elliptic and A_h has a strongly invariant multicone $\mathcal{C}$ such that $A\mathcal{C} = \mathcal{C}$ for all $A \in \mathsf{A}_e$.

Characterization of equilibrium states

- Further, let $f\colon \Sigma \to \mathbb{R}$ be continous and $\Psi = (\sum_{k=0}^{n-1} f \circ \sigma^k)_{n\in\mathbb{N}}$. Recall that a probability measure μ is a *Gibbs measure* for f if there exists a constant $C \geq 1$ such that

$$C^{-1} \leq \frac{\mu([\mathtt{i}])}{\exp\bigl(-nP(\Psi) + \sum_{k=0}^{n-1} f(\sigma^k \mathtt{j})\bigr)} \leq C$$

for all $\mathtt{i} \in \Sigma_n$, $\mathtt{j} \in [\mathtt{i}]$, and $n \in \mathbb{N}$.

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If $\mathsf{A} \in GL_2(\mathbb{R})^N$ is irreducible and μ is the equilibrium state for Φ^s, then the following are equivalent:

(1) μ is a Gibbs measure for a Hölder continuous function,

(2) A has a strongly invariant multicone or A is strongly elliptic.

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- A probability measure μ on Σ is *Bernoulli* if
$$\mu([\mathtt{i}\,\mathtt{j}]) = \mu([\mathtt{i}])\mu([\mathtt{j}])$$
for all $\mathtt{i}, \mathtt{j} \in \Sigma_*$. In other words, μ is Bernoulli if there exists a probability vector $(p_1, \ldots, p_N)$ such that
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If $\mathsf{A} \in GL_2(\mathbb{R})^N$ is irreducible and μ is the equilibrium state for Φ^s, then the following are equivalent:

(1) μ is a Bernoulli measure,
(2) A is strongly elliptic.

Characterization of equilibrium states

- It can happen that an equilibrium state for Φ^s is a Gibbs measure for some Hölder-continuous potential, but is not a Bernoulli measure: Choose two positive matrices

$$A_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then (A_1, A_2) is irreducible and has a strongly invariant multicone (i.e. the union of the first and third quadrants).

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- It can happen that an equilibrium state for Φ^s is a quasi-Bernoulli measure, but is not a Gibbs measure for any Hölder-continuous potential: Let A_1 and A_2 be as above. Then (A_1, A_2, I) is irreducible and has an invariant multicone (i.e. the union of the first and third quadrants).

Characterization of equilibrium states

- It can happen that an equilibrium state for Φ^s is a Gibbs-type measure for Φ^s, but is not a quasi-Bernoulli measure: Choose two matrices
$$A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad A_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
Then (A_3, A_4) is irreducible, has no invariant multicone, and does not contain only elliptic matrices.

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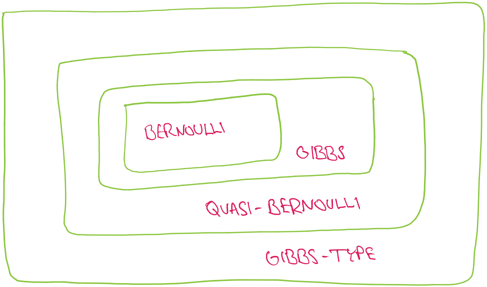

Another subadditive potential

- Recall that the singular values $\|A\| = \alpha_1(A) \geq \cdots \geq \alpha_d(A) > 0$ of $A \in GL_d(\mathbb{R})$ are the square roots of the non-negative real eigenvalues of the positive semidefinite matrix $A^T A$.

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- If $0 \leq s \leq d$ and $k \leq s < k+1$, then the *singular value function* of $A \in GL_d(\mathbb{R})$ with parameter s is
$$\begin{aligned}\varphi^s(A) &= \alpha_1(A) \cdots \alpha_k(A) \alpha_{k+1}(A)^{s-k} \\ &= \|A^{\wedge k}\|^{k+1-s} \|A^{\wedge (k+1)}\|^{s-k}.\end{aligned}$$
It follows that $\varphi^s(AB) \leq \varphi^s(A)\varphi^s(B)$ for all $A, B \in GL_d(\mathbb{R})$, and hence, the sequence $\hat{\Phi}^s = (\log \varphi^s(A_{\mathtt{i}|_n}))_{n \in \mathbb{N}}$ is subadditive.

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- Note that all the previous results hold for $\hat{\Phi}^s$ when $d = 2$ since

$$\varphi^s(A) = \begin{cases} \|A\|^s, & \text{if } 0 < s < 1, \\ |\det(A)|^{s-(d-1)} \|A^{\wedge (d-1)}\|^{d-s}, & \text{if } d-1 < s < d. \end{cases}$$

Equilibrium states for singular value function

Theorem (Feng & K., 2011)

If $0 < s < 2$ and $\mathsf{A} \in GL_2(\mathbb{R})$, then the maximum possible number of distinct ergodic equilibrium states for $\hat{\Phi}^s$ is 2 and every equilibrium state is fully supported.

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Theorem (Morris & K., 2018)

If $0 < s < 3$ and $\mathsf{A} \in GL_3(\mathbb{R})^N$, then the maximum possible number of distinct ergodic equilibrium states for $\hat{\Phi}^s$ is 6, if $1 < s < 2$, and 3, if otherwise, and every equilibrium state is fully supported.

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- If $s \in (0,1) \cup (2,3)$, then the above result follows from Feng & K. (2011). The case $1 < s < 2$ divides into three further subcases which are all studied by completely different methods.

Equilibrium states for singular value function

Theorem (Bochi & Morris, to appear)

If $0 < s < d$ and $\mathsf{A} \in GL_d(\mathbb{R})^N$, then there are at most finitely many distinct ergodic equilibrium states for $\hat{\Phi}^s$, and every equilibrium state is fully supported.

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If $0 < s < d$ and $\mathsf{A} \in GL_d(\mathbb{R})^N$, then there are at most finitely many distinct ergodic equilibrium states for $\hat{\Phi}^s$, and every equilibrium state is fully supported.

Question

What is the maximum possible number of distinct ergodic equilibrium states for Φ^s in higher dimensions?

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Theorem (Li & K., 2017)

The set $\{\mathsf{A} \in GL_d(\mathbb{R}) : \text{equilibrium state for } \hat{\Phi}^s \text{ is not unique}\}$ is contained in a finite union of $(d^2N - 1)$-dimensional algebraic varieties.

Self-affine sets

- If $f_1, \ldots, f_N \colon \mathbb{R}^d \to \mathbb{R}^d$ are affine, i.e. $f_i(x) = A_i x + v_i$, where $A_i \in GL_d(\mathbb{R})$ and $v_i \in \mathbb{R}^d$, then the tuple $(f_1, \ldots, f_N)$ is called an *affine IFS* and there exists a nonempty compact set $E \subset \mathbb{R}^d$ such that
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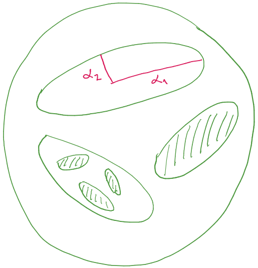
$$E = \bigcup_{i=1}^{N} f_i(E).$$
- A central problem is to calculate or estimate the dimension of the *self-affine* set E.



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- In the special case, where each A_i is a scalar multiple of an isometry, i.e. $f_i(x) = r_i O_i x + v_i$, then $(f_1, \ldots, f_N)$ is called a *similitude IFS* and the set E *self-similar*.

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  The most recent advance in the Erdős' problem is by Shmerkin (2014). His proof used self-similar sets.

  Jordan & Sahlsten (2016) solved the Salem's problem by using IFSs.

Dimension theory of self-similar sets

- If $(f_1, \ldots, f_N)$ is a similitude IFS satisfying the SSC (i.e. $f_i(E) \cap f_j(E) = \emptyset$ for $i \neq j$), then it is a classical result that

$$\dim_{\mathrm{H}}(E) = s,$$

where $\sum_{i=1}^{N} r_i^s = 1$ or, equivalently, $P(\Phi^s) = 0$, where $\Phi^s = (\log r_{\mathtt{i}|_n}^s)_{n\in\mathbb{N}}$ and $r_{\mathtt{i}|_n} = r_{i_1} \cdots r_{i_n}$ for all $\mathtt{i} = i_1 i_2 \cdots \in \Sigma$.

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- Without separation the question is difficult. Write $f_{\mathtt{i}} = f_{i_1} \circ \cdots \circ f_{i_n}$ for all $\mathtt{i} = i_1 \cdots i_n \in \Sigma_n$ and $n \in \mathbb{N}$, and say that E has *exact overlaps* if $f_{\mathtt{i}} = f_{\mathtt{j}}$ for some $\mathtt{i} \neq \mathtt{j}$.

Theorem (Hochman, 2014)

If the similitude IFS in the real line is defined by algebraic parameters, then the associated self-similar set E either has exact overlaps or $\dim_{\mathrm{H}}(E) = \min\{1, s\}$, where $P(\Phi^s) = 0$.

Dimension theory of self-affine sets

- Although it is easy to find affine IFSs $(f_1, \ldots, f_N)$ satisfying the SSC such that $\dim_{\mathrm{H}}(E) < s$, where $P(\hat{\Phi}^s) = 0$, it is still expected that this s, denoted by $\dim_{\mathrm{aff}}(\mathsf{A})$, gives the dimension for a large class of self-affine sets.

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- Recall that the *singular value function* of A with parameter s is

$$\varphi^s(A) = \alpha_1(A) \cdots \alpha_{\lfloor s \rfloor}(A) \alpha_{\lceil s \rceil}(A)^{s - \lfloor s \rfloor},$$

where $\alpha_i(A)$ is the length of the ith semiaxis of $A(B(0,1))$.

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- For example, if $d = 2$ and $1 \le s < 2$, then

$$\varphi^s(A) = \alpha_1(A)\alpha_2(A)^{s-1} = \frac{\alpha_1(A)}{\alpha_2(A)} \alpha_2(A)^s,$$

where $\frac{\alpha_1(A)}{\alpha_2(A)}$ roughly tells how many balls of radius $\alpha_2(A)$ are needed to cover $A(B(0,1))$.

Dimension theory of self-affine sets

- Assuming $d = 2$ and $1 \le s < 2$, we have
$$\mathcal{H}^s(E) \lesssim \lim_{n\to\infty} \sum_{\mathtt{i}\in\Sigma_n} \frac{\alpha_1(A_{\mathtt{i}})}{\alpha_2(A_{\mathtt{i}})} \alpha_2(A_{\mathtt{i}})^s.$$
Since
$$P(\hat{\Phi}^s) = \lim_{n\to\infty} \tfrac{1}{n} \log \sum_{\mathtt{i}\in\Sigma_n} \frac{\alpha_1(A_{\mathtt{i}})}{\alpha_2(A_{\mathtt{i}})} \alpha_2(A_{\mathtt{i}})^s$$
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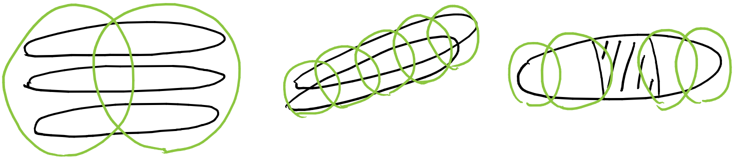
is strictly decreasing in s, we see that $\dim_{\mathrm{H}}(E) \le \dim_{\mathrm{aff}}(\mathsf{A})$.
- If $\dim_{\mathrm{H}}(E) < \dim_{\mathrm{aff}}(\mathsf{A})$, then the covers obtained from the ellipses are not optimal:



Dimension theory of self-affine sets

- Thus far the dimension theory of self-affine sets has focused on specific subclasses. The problem of finding the dimension can be make more tractable either by assuming some randomness in the defining IFS or by imposing special relations between the affine maps.

Dimension theory of self-affine sets

- Thus far the dimension theory of self-affine sets has focused on specific subclasses. The problem of finding the dimension can be make more tractable either by assuming some randomness in the defining IFS or by imposing special relations between the affine maps.
- The following theorem is the first dimension result valid for all (planar) self-affine sets.

Theorem (Bárány & K., 2017)

If μ is a Bernoulli measure on Σ and $\pi\mu$ is a self-affine measure on a self-affine set E, then $\pi\mu$ is exact-dimensional and satisfies the so-called Ledrappier-Young formula.

Here π is the canonical projection $\Sigma \to E$.

Dimension theory of self-affine sets

- A classical dimension result for self-affine sets is due to Falconer. It guarantees that for a random choice of translation vectors the covers are optimal.

Theorem (Falconer, 1988)

If $\mathsf{A} = (A_1, \ldots, A_N) \in GL_d(\mathbb{R})^N$ satisfies $\|A_i\| < 1/2$ for all i, then

$$\dim_{\mathrm{H}}(E_{\mathsf{A},\mathsf{v}}) = \dim_{\mathrm{aff}}(\mathsf{A})$$

for $\mathcal{L}^{dN}$-almost all $\mathsf{v} = (v_1, \ldots, v_N) \in (\mathbb{R}^d)^N$.

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- The proof of the result follows from the existence of an equilibrium state for $\hat{\Phi}^s$ and a transversality argument.

  Relying also on the Ledrappier-Young formula, one can show an orthogonal version for Falconer's result.

Dimension theory of self-affine sets

Theorem (Bárány & Koivusalo & K., to appear)

If $\mathsf{v} = (v_1, \ldots, v_N) \in (\mathbb{R}^d)^N$ is such that $v_i \neq v_j$ for $i \neq j$, then

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for $\mathcal{L}^{d^2N}$-almost all $\mathsf{A} = (A_1, \ldots, A_N) \in \mathcal{A}_{\mathsf{v}} \subset GL_d(\mathbb{R})^N$.

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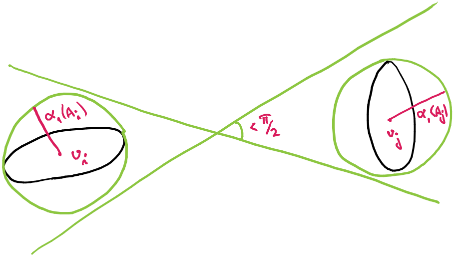
In $\mathbb{R}^2$, the set $\mathcal{A}_{\mathsf{v}} \subset GL_2(\mathbb{R})^N$ is the collection of matrix tuples satisfying the following:



Dimension theory of self-affine sets

- The planar version of the previous theorem is generalized by the following theorem.

Theorem (Bárány & Hochman & Rapaport, preprint)

If $(f_1, \ldots, f_N)$ is a planar affine IFS satisfying the SSC such that the associated matrix tuple $\mathsf{A} \in GL_2(\mathbb{R})^N$ is strongly irreducible, then

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- Very roughly speaking, the proof follows from Hochman's dimension result for self-similar sets in the real line, the Ledrappier-Young formula, and the existence of the equilibrium state for $\hat{\Phi}^s$.

Recent literature

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- Multifractal formalism on self-affine sets: Reeve & K. (2014), Jordan & Rams (2015), Fraser & Kempton (2018), Bárány & Jordan & Rams & K. (preprint).

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