Comparing throughputs and fairness in slotted and non-slotted CSMA

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(Joint work with Peter van de Ven)

Non-slotted CSMA (Carrier Sense Multiple Access): Model description

- n transmitters/receivers on a line
- k-hop interference: a transmitter may activate only if no neighbouring transmitters within k-hop range are active
- All transmission times are exponential with rate 1
- If no neighbouring nodes are active, then each node activates after an exponential time with rate σ
- Now: all nodes are saturated, i.e. always have a message to transmit

Non-slotted CSMA: throughputs

We assume that $k = 1$. The throughput of node i

$$\theta_i = \sigma \frac{Z_{i-2} Z_{n-i-1}}{Z_n},$$

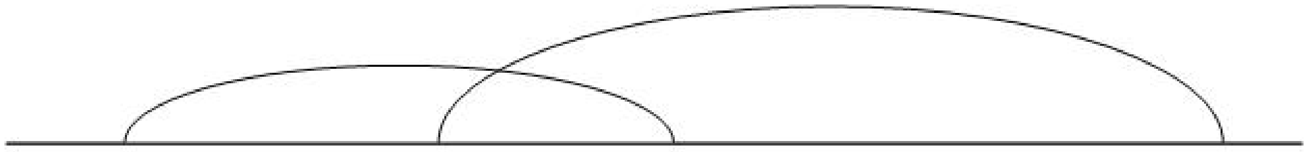
where $Z_i = 1$ for $i \leq 0$ and

$$Z_i = \frac{1}{\sqrt{1+4\sigma}} \left(\left(\frac{1+\sqrt{1+4\sigma}}{2} \right)^{i+2} - \left(\frac{1-\sqrt{1+4\sigma}}{2} \right)^{i+2} \right)$$

otherwise (Pinsky, Yemini, 1986).

Early days of multiple access..

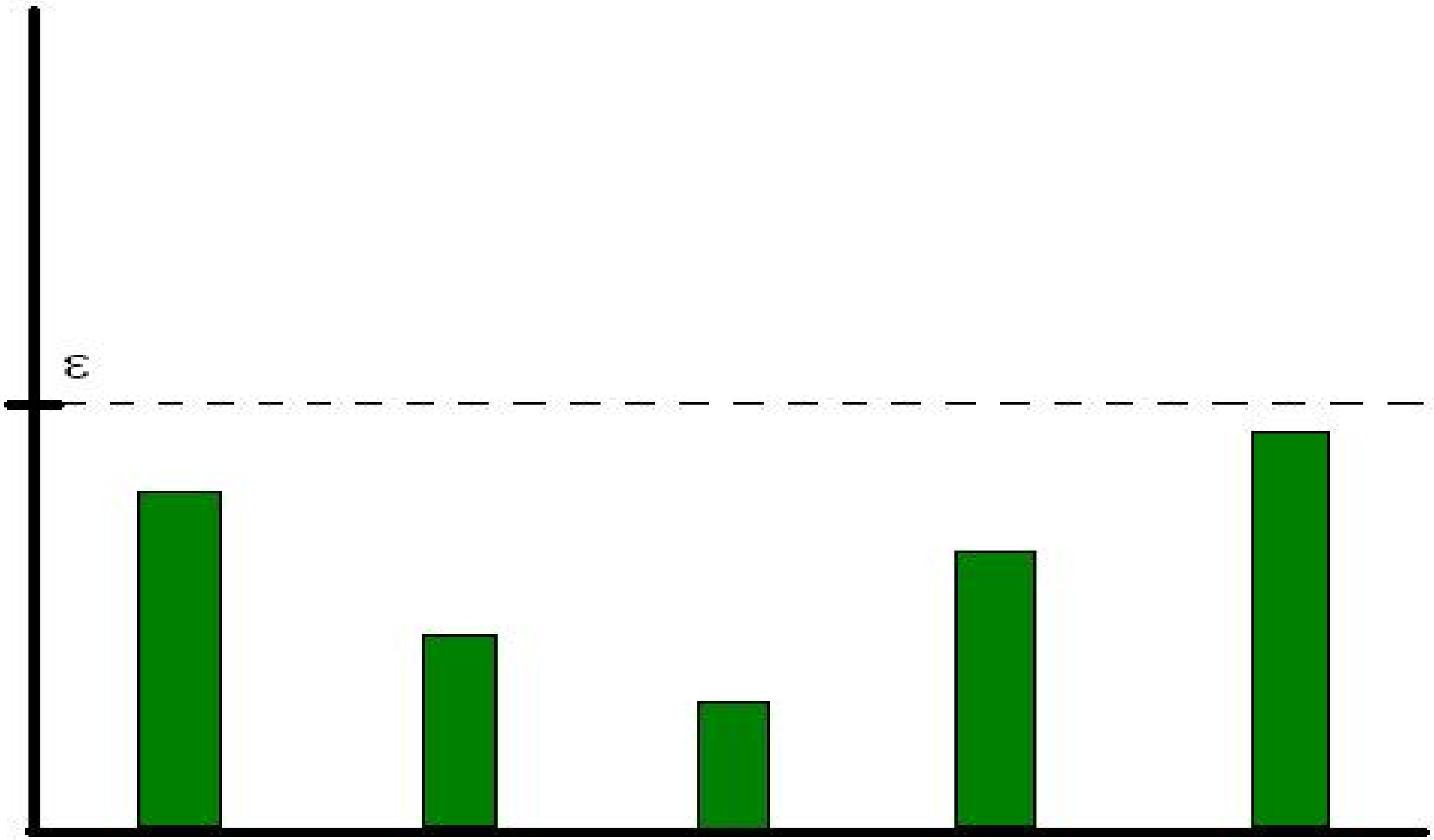
- ALOHA (Abramson, 1970): If you have data, send it. If the message is not received well, wait a ”random time” and send it again. Throughput $e^{-1}/2$
- Slotted ALOHA (Roberts, 1972): All transmission times are equal to 1, transmissions may only be attempted at the beginning of a time slot, and every transmission is attempted with a certain probability. Throughput e^{-1}
- Reason: less ”wasted” time



Slotted CSMA: model description (Durvy and Thiran, 2006)

- Transmission attempts are only possible at time instants $1, 2, \ldots$
- Transmission times are equal to 1
- At the beginning of a time slot, each node is given a priority number, collection of priorities at any given time slot is a permutation of numbers $1, 2, \ldots, n$. The node with the highest priority starts transmitting, thus blocking its neighbours. Then the node with the highest priority among the remaining nodes starts transmitting, etc.
- At the next slot everything is repeated, independently of all previous slots

Slotted CSMA: how it could be implemented



Slotted CSMA: total throughput of the system

If we call E_n the (total) throughput of a system with n nodes, then

$$E_n = 1 + \frac{2}{n} E_{n-2} + \sum_{k=2}^{n-1} \frac{1}{n} (E_{k-2} + E_{n-k-1})$$

We can rewrite the previous equation as

$$E_n = 1 + \frac{2}{n} \sum_{k=1}^{n} E_{k-2}.$$

Introduce now the function

$$\phi(\rho) = \sum_{n=1}^{\infty} E_n \rho^n.$$

Slotted CSMA: total throughput of the system

In order to find $\phi(\rho)$, compute its derivative:

$$\begin{aligned}
\phi'(\rho) = \sum_{n=1}^{\infty} nE_n\rho^{n-1} &= \sum_{n=1}^{\infty} n\left(1+\frac{2}{n}\sum_{k=1}^{n}E_{k-2}\right)\rho^{n-1} \\
&= \sum_{n=1}^{\infty} n\rho^{n-1} + 2\sum_{n=1}^{\infty}\sum_{k=1}^{n}E_{k-2}\rho^{n-1} \\
&= \frac{1}{(1-\rho)^2} + 2\sum_{k=1}^{\infty}E_{k-2}\sum_{n=k}^{\infty}\rho^{n-1} \\
&= \frac{1}{(1-\rho)^2} + \frac{2}{1-\rho}\sum_{k=1}^{\infty}E_{k-2}\rho^{k-1} = \frac{1}{(1-\rho)^2} + \frac{2\rho}{1-\rho}\phi(\rho).
\end{aligned}$$

Thus, we have a first-order differential equation with the initial condition $\phi(0) = 0$. Its solution is

$$\phi(\rho) = \frac{1-e^{-2\rho}}{2(1-\rho)^2}.$$

Slotted CSMA: total throughput of the system

Now use $1 - e^{-2\rho} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n!} \rho^n$ and $(1-\rho)^{-2} = \sum_{n=1}^{\infty} n\rho^{n-1} = \sum_{n=0}^{\infty} (n+1)\rho^n$ to conclude that

$$\phi(\rho) = \frac{1}{2} \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} (-1)^{k+1} \frac{2^k}{k!} (n-k+1) \right) \rho^n,$$

and hence

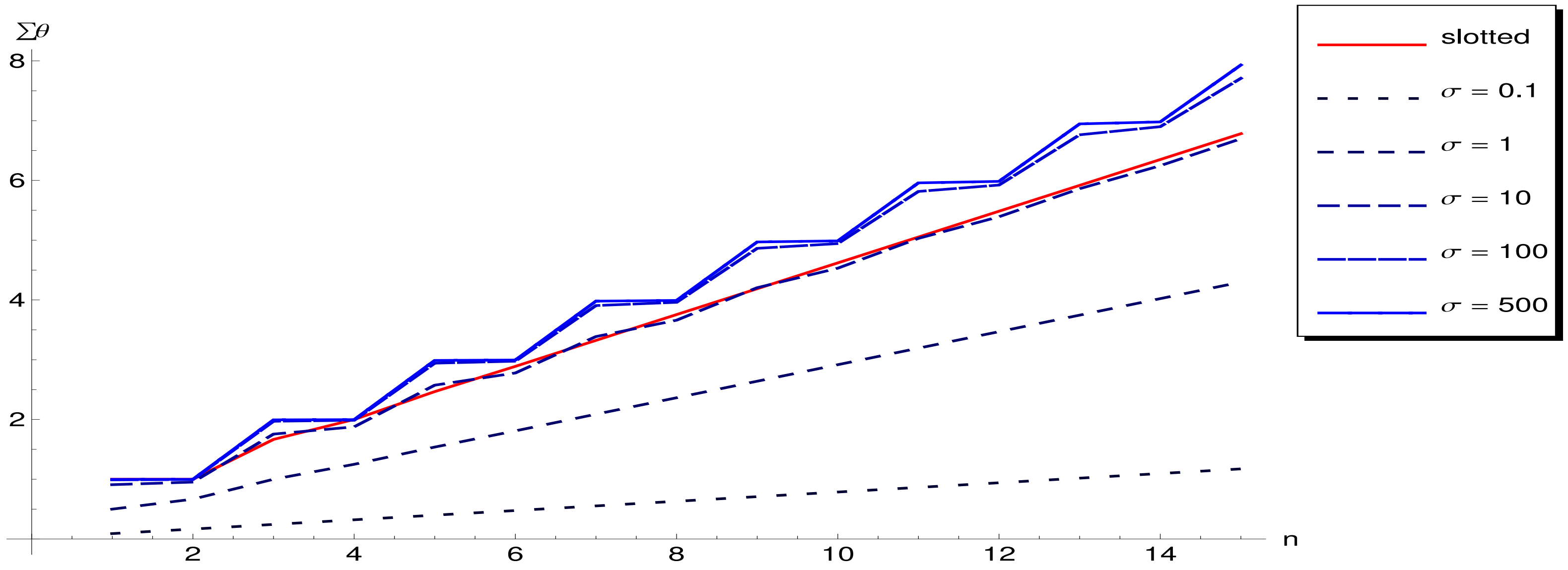
$$E_n = \frac{1}{2} \sum_{k=1}^{n} (-1)^{k+1} \frac{2^k}{k!} (n-k+1).$$

Interesting:

$$\frac{E_n}{n} = \frac{1}{2} \sum_{k=1}^{n} (-1)^{k+1} \frac{2^k}{k!} - \frac{1}{2n} \sum_{k=1}^{n} (-1)^{k+1} \frac{2^k}{k!} (k-1) \to \frac{1}{2} \left(1 - e^{-2}\right)$$

It is known, however, that the total throughput of a non-slotted system tends to the maximal possible one (in this case $\lceil n/2 \rceil$) when σ becomes large.

Total throughputs: comparison



Question: From which value of σ does non-slotted outperform slotted?

Individual throughputs in the slotted case

Denote by $T_i(n)$ the throughput of node i in a system with n nodes. Then

$$T_1(n) = \frac{1}{n} + \frac{1}{n}\sum_{i=3}^{n} T_1(i-2) = \frac{1}{n} + \frac{1}{n}\sum_{i=1}^{n-2} T_1(i).$$

Introduce

$$\psi_1(\rho) = \sum_{n=1}^{\infty} T_1(n)\rho^n.$$

Following arguments similar to those used to obtain total throughputs, we get

$$\psi_1'(\rho) = \frac{1}{1-\rho} + \frac{\rho}{1-\rho}\psi_1(\rho)$$

with $\psi_1(0) = 0$, solve it, write Taylor expansions and eventually obtain

$$T_1(n) = \sum_{j=1}^{n} (-1)^{j+1}\frac{1}{j!}.$$

Individual throughputs in the slotted case

Similarly with $T_2(n)$:

$$T_2(n) = \frac{1}{n} + \frac{1}{n}\sum_{i=4}^{n} T_2(i-2) = \frac{1}{n} + \frac{1}{n}\sum_{i=2}^{n-2} T_2(i),$$

Defining now $\psi_2(\rho) = \sum_{n=2}^{\infty} T_2(n)\rho^n$ and repeating the arguments used above, we get that

$$\psi_2'(\rho) = \frac{\rho}{1-\rho} + \frac{\rho}{1-\rho}\psi_2(\rho).$$

Solve this equation with $\psi_2(0) = 0$ and use Taylor expansions to get

$$T_2(n) = \sum_{j=2}^{n} (-1)^j \frac{1}{j!}.$$

Individual throughputs in the slotted case

Slightly more complicated for $T_i(n)$ with $i \geq 3$:

$$T_i(n) = \frac{1}{n} + \frac{1}{n}\sum_{j=1}^{i-2} T_{i-j-1}(n-j-1) + \frac{1}{n}\sum_{j=i}^{n-2} T_i(j),$$

and the differential equation for $\psi_i(\rho) = \sum\limits_{n=i}^{\infty} T_i(n)\rho^i$ has the form

$$\psi_i'(\rho) = \sum_{j=1}^{i-2} \rho^j \psi_{i-j-1}(\rho) + \frac{\rho^{i-1}}{1-\rho} + \frac{\rho}{1-\rho}\psi_i(\rho).$$

To solve this system of equations, introduce

$$\nu(\rho, s) = \sum_{i=1}^{\infty} \psi_i(\rho) s^i.$$

Individual throughputs in the slotted case

We get the following differential equation:

$$\frac{d\nu(\rho,s)}{d\rho} = \left(\frac{\rho s^2}{1-\rho s} + \frac{\rho}{1-\rho}\right)\nu(\rho,s) + \frac{s}{(1-\rho)(1-\rho s)}$$

with $\nu(s,0) = 0$. This gives us

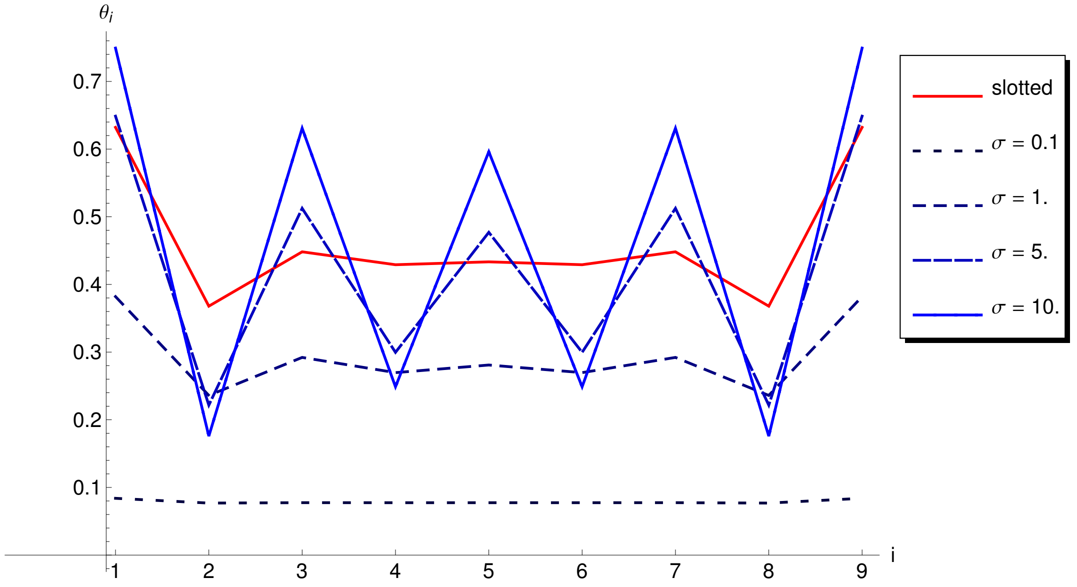
$$\nu(\rho,s) = \frac{s}{s+1}\frac{1}{1-\rho}\frac{1}{1-\rho s}\left(1 - e^{-\rho(s+1)}\right)$$

and eventually

$$\psi_i(\rho) = \frac{1}{1-\rho^2}\left[\rho^i + (-1)^{i+1} - e^{-\rho}\rho^i a_i + (-1)^i e^{-\rho}\sum_{k=0}^{i-1}\frac{\rho^k}{k!}\right],$$

which gives explicit formulas for $T_i(n)$...

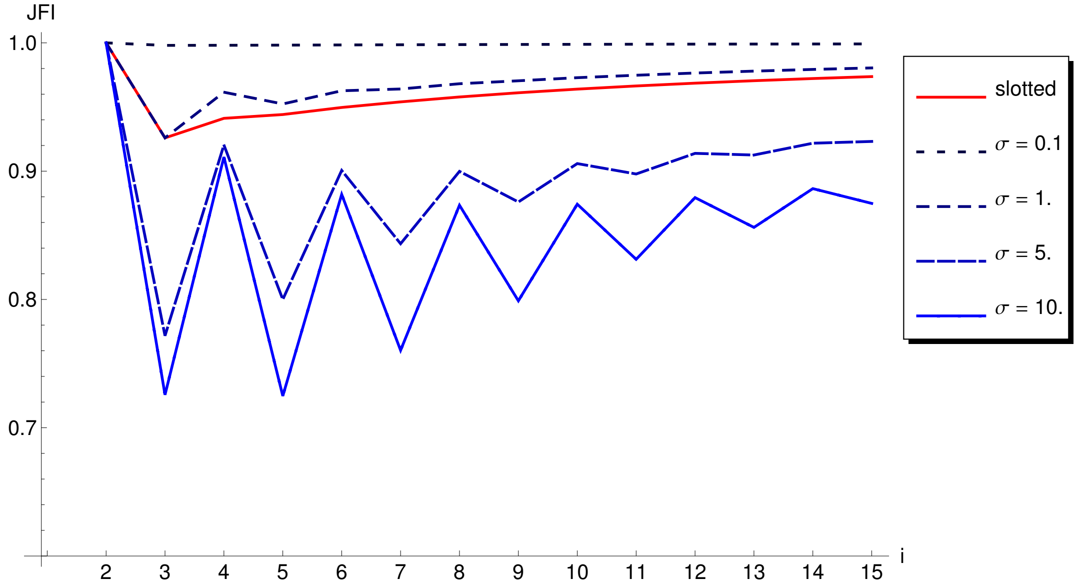
Comparison of individual throughputs



Fairness

Jain's fairness index:

$$\frac{\left(\sum_{i=1}^{n}\theta_i\right)^2}{n\sum_{i=1}^{n}\theta_i^2}.$$



Question: Does it converge to 1?

Question: When does slotted start outperforming non-slotted?

Non-saturated systems

- Now only the first node is saturated
- The transmissions have to travel through the system from left to right, until they are transmitted by the right-most node, upon which they leave the system
- For the non-slotted case, we assume that $\sigma = \infty$

Non-slotted case, $n = 3$, $k = 1$

- Node 2 gets saturated
- The queue at node 3 is stable
- The state of the system is described by a two-dimensional Markov Chain $\{X(t), Y_3(t)\}$, where $X(t) \in \{0, 1\}$ shows whether outer nodes are active or not, and $Y_3(t)$ is the number of packets in node 3.
- Balance equations yield for stationary probabilities $\pi(x, y) = \lim_{t\to\infty} \mathsf{P}\left((X(t), Y_3(t)) = (x, y)\right)$ (Borst, Denteneer, Hiertz, van de Ven)

$$\pi(0, y) = \frac{1}{5}\left(\frac{1}{3}\right)^{y}$$

and

$$\pi(1, 0) = \frac{2}{5}, \quad \pi(1, y) = \frac{1}{5}\left(\frac{1}{3}\right)^{y-1} \quad \text{for} \quad y \geq 1.$$

Non-slotted case, $n = 3$, $k = 1$

The throughput is then equal to the fraction of time node 3 is active:

$$\sum_{y=1}^{\infty} \pi(1, y) = \frac{3}{10}.$$

Or, similarly, the throughput is the fraction of time node 2 is active:

$$\sum_{y=0}^{\infty} \pi(0, y) = \frac{3}{10}$$

Slotted case, $n = 3$, $k = 1$

- Node 2 is saturated
- The queue at node 3 is stable
- The state of the system is described by a Markov Chain $\{Y_3(t)\}$
- For stationary probabilities $\pi(y) = \lim_{t\to\infty} \mathrm{P}\left(Y_3(t) = y\right)$ we have

$$\pi_0 = \frac{1}{2}\pi_0 + \frac{2}{3}\pi_1, \quad \pi_1 = \frac{1}{2}\pi_0 + \frac{2}{3}\pi_2$$

  and

$$\pi_i = \frac{1}{3}\pi_{i-1} + \frac{2}{3}\pi_{i+1} \quad \text{for} \quad i \geq 2.$$

- These equations lead to $\pi_0 = \dfrac{2}{5}$ and $\pi_i = \dfrac{3}{2^{i+1}}\pi_0$ for $i \geq 1$.
- The throughput is then equal to $\dfrac{2}{3}(1-\pi_0) = \dfrac{1}{2}\pi_0 + \dfrac{1}{3}(1-\pi_0) = \dfrac{2}{5}$.

Non-slotted AND slotted case, $n \leq 2k + 1$

- Nodes from 2 to $k + 1$ get saturated
- Queues of nodes $k + 2$ up to n are stable
- Thus the time may be divided into cycles that run between the epochs when nodes $k + 2$ to n are empty
- The throughput is equal to the ratio of the average number of transmissions made by node $k + 1$ over a typical cycle to the average duration of a typical cycle

Both cases, $n = 2k + 1$, bounds for the throughput

In the non-slotted case

$$\frac{1}{2k+2} < T < \frac{1}{2k+1}$$

In the slotted case

$$T > \frac{1}{2k+1}$$

Remark: similar bounds hold for all $n \leq 2k + 1$, and we can prove them. The exact same bounds seem to hold for all $n \geq 2k + 1$, but we can't prove them yet.

(Some) possible research directions

- Stability analysis in the case of randomly generated messages
- Algorithms that make the system stable even if the origin is saturated $\Rightarrow$ tradeoff between stability and throughput
- Other graphs, multiple routes