Evolutionary Dynamics, Equilibrium Selection and Network Structure

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(joint work with A. Montanari)

Outline

- Motivating question: diffusion of behavior or technology in a social network

- Ising model, Glauber dynamics, and meta stability

- Interpretation of the results in the context of social networks

Models for the spread of behavior or technology

- Epidemic or cascade models (contact process e.g. Liggett, Kleinberg)

- Dynamics of a game
(individual strategy and social structure)

Coordination games

- Players: nodes in graph G(V,E)
- Action space

$$\underline{x} = \{x_i : i \in V\},\ x_i \in \{+1, -1\}$$

- Payoff matrix
  - A > D and B > C

	$+$	$-$
$+$	A	C
$-$	D	B

Example

- Choosing a cell phone provider

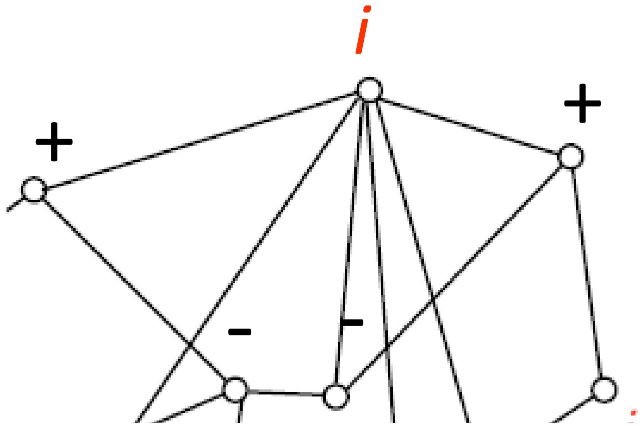
  + : AT&T

  − : Verizon

	+	−
+	6	2
−	2	5

- Payoff of *i* in the network

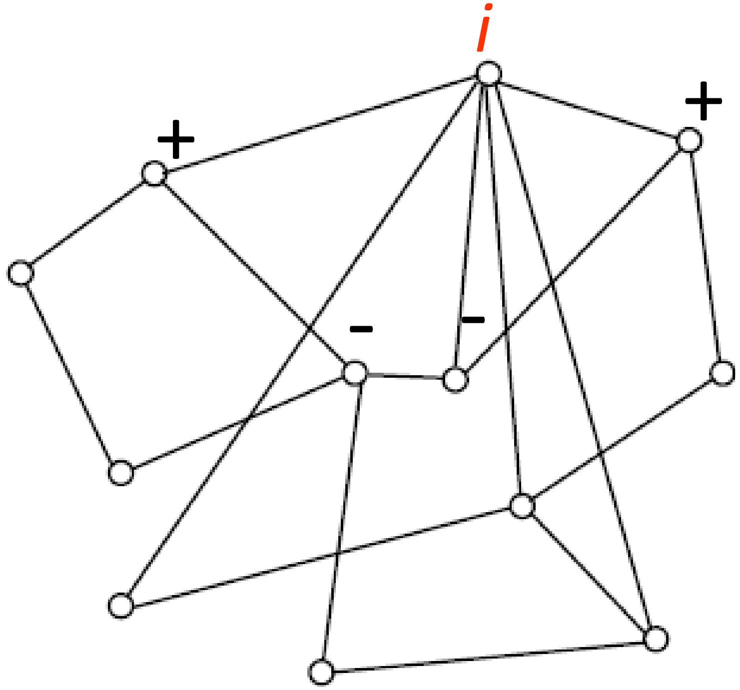
$$+ : 6N_{+}(i) + 2N_{-}(i)$$
$$- : 2N_{+}(i) + 5N_{-}(i)$$



Best response dynamics

- For every *i* there is an h_i such that best response is

$$\operatorname{sign}\Big(h_i + \sum_{j \in \partial i} x_j(t)\Big)$$



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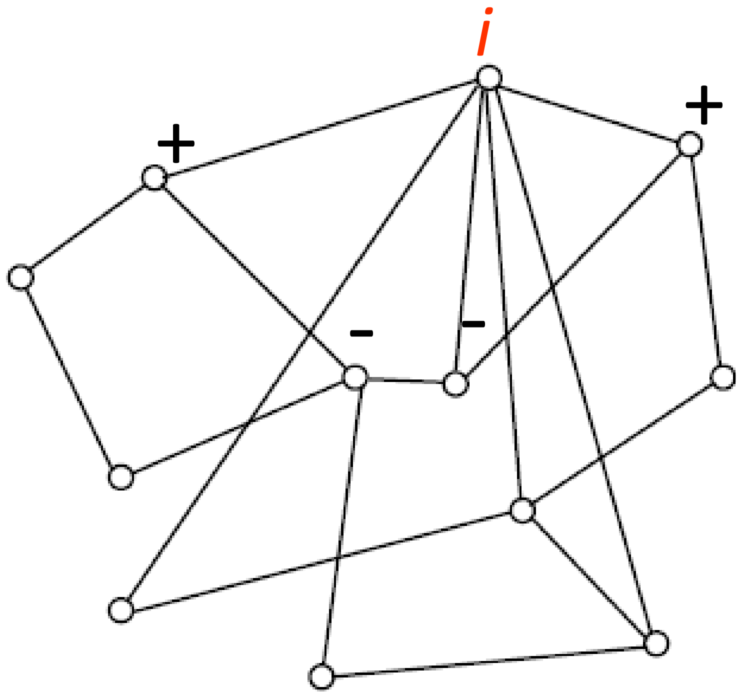
$$h_i = \frac{a-d-b+c}{a-d+b-c}|\partial i|$$



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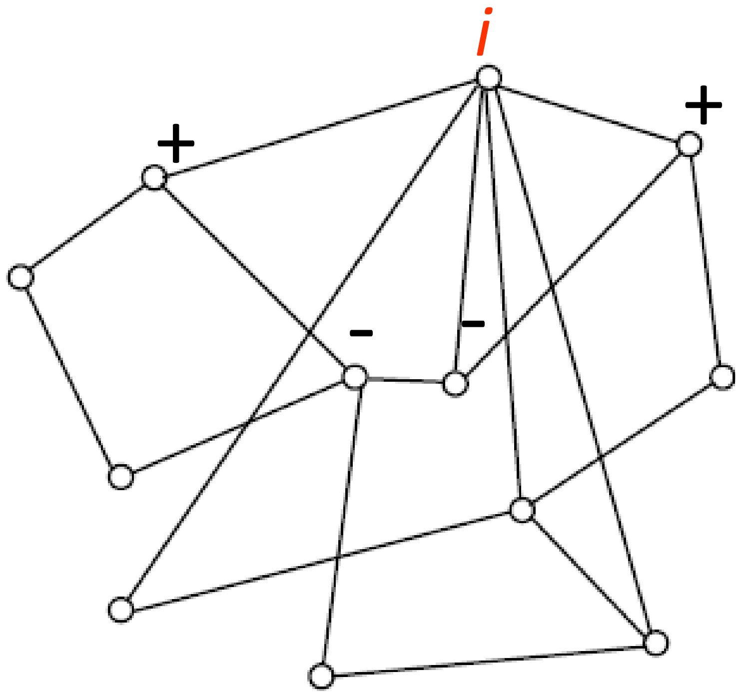
wlog assume $h_i > 0$



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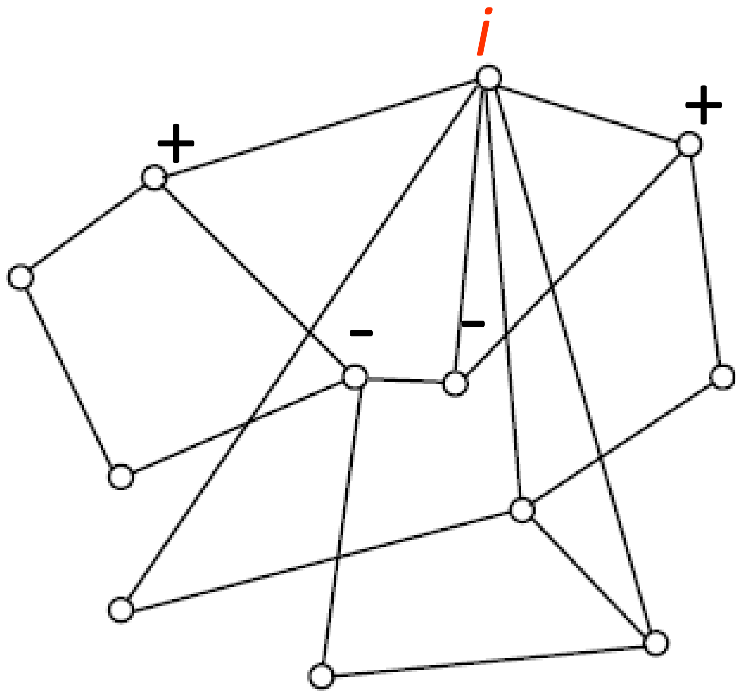


- Best response dynamics increases the potential function

$$H(\underline{x}) = \sum_{(i,j) \in E} x_i x_j + \sum_{i \in V} h_i x_i$$

Example

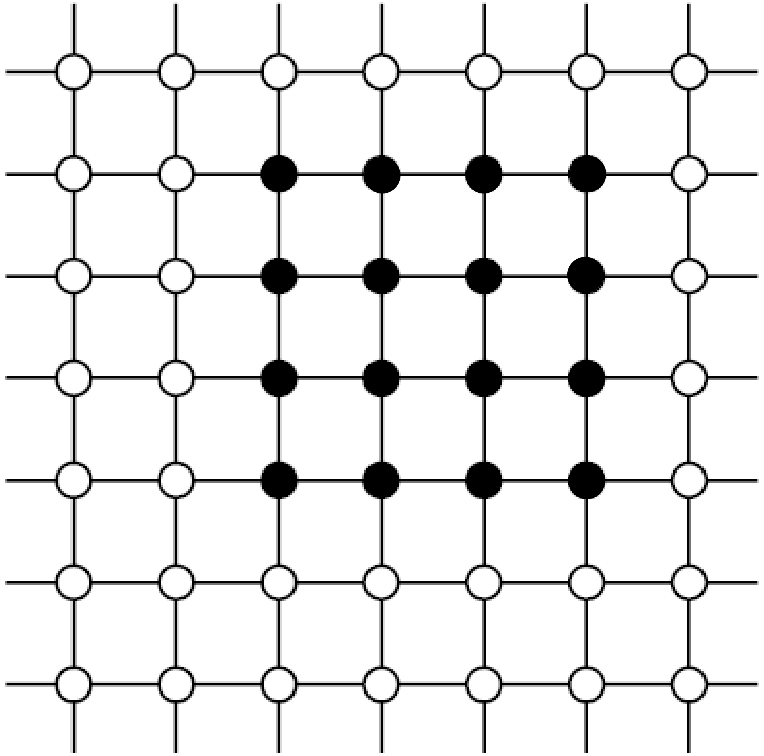
Suppose h_i = 0.1 for all i

Best response dynamics: stays at this equilibrium

Allow for some noise:

the + area expands to the whole graph

Kandori, Mailath, Rob 93
also physics (nucleation)



The effect of graph structure

- $\tau_+(G)$: time to hit all +

- Ellison (1993) : if the probability of mistake is ^{2}

Small no of neighbors

$$\text{If } G = (\text{LINE})^k, \quad \text{then } \tau_+(G) = \epsilon^{-\Theta(1)}.$$
$$\text{If } G = K_n, \quad \text{then } \tau_+(G) = \epsilon^{-\Theta(n)}.$$

Global interaction

Our goal

- Find the graph theoretic property that captures the rate of convergence

- Demonstrate on specific graph sequences
  - Expanders
  - Finite range graphs
  - Small-worlds

The dynamics (definition)

Asynchronous: strategy revision at arrival of a Poisson clock

$$\Pr(x_i = y|\partial i) = \exp\left\{\beta y(h_i + \sum_{j\in\partial i} x_j(t))\right\}$$

Glauber dynamics for the Ising model

Reversible: with the stationary distribution

$$\mu(\underline{x}) \propto \exp\left\{\beta H(\underline{x})\right\} = \exp\left\{\beta \sum_{(i,j)\in E} x_i x_j + \beta \sum_{i\in V} h_i x_i\right\}$$

The measure concentrates around +1 as ¯ $\longrightarrow$1

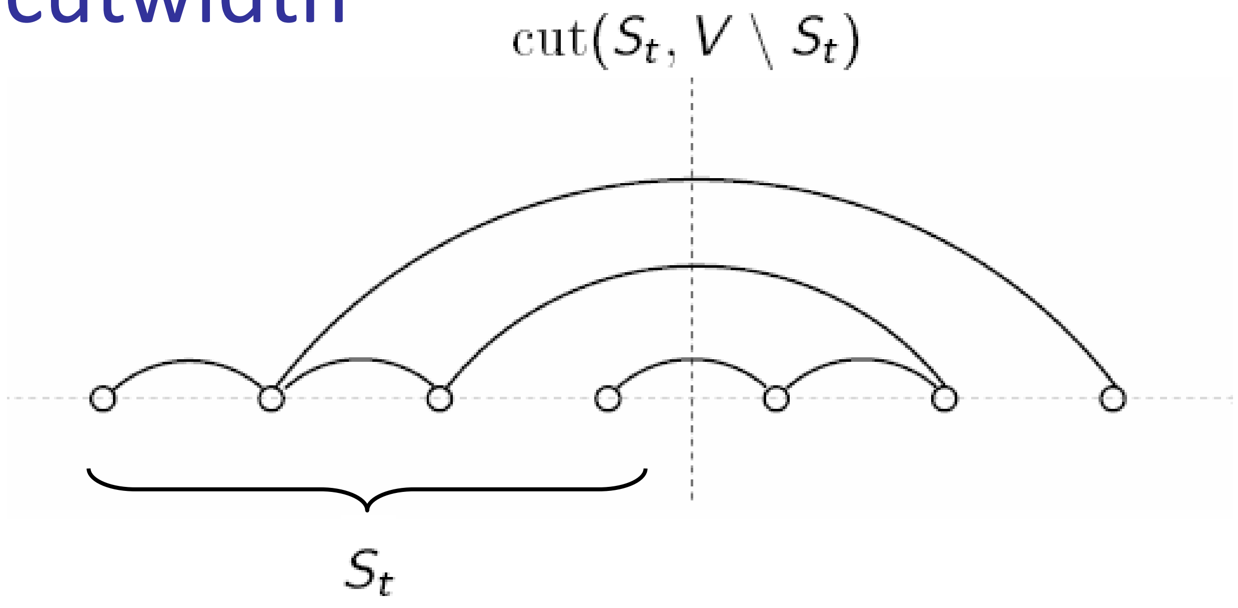
The question

Meta-stability: typical hitting time of + from the worst starting point

$$\tau_+(G) \equiv \sup_{\underline{x}} \inf \left\{ t \geq 0 : \mathbb{P}_{\epsilon}^{\underline{x}}\{T_+ \geq t\} \leq e^{-1} \right\}$$

Studied in mathematical physics for 2,3D grids
(c.f. Neves & Schonman 92, Bovier & Manzo 02)

Tilted cutwidth



$$\Gamma(G;\underline{h}) \equiv \min_{S:\emptyset\to V} \max_{t\le n} \left[\mathrm{cut}(S_t, V\setminus S_t) - |S_t|_h\right]$$

$$|S|_h \equiv \sum_{i\in S} h_i$$

Note that cutwidth is simply $\Gamma(G;0)$

Proposition: for large enough ¯

$$\tau_+(G) = \exp\{\beta\Gamma_*(G;\underline{h}) + o(\beta)\}\,,$$

where

$$\Gamma_*(G;\underline{h}) = \max_{F\subseteq G}\Gamma(F;\underline{h}^F)$$

$$h_i^F = h_i + |\partial i|_{G\setminus F}.$$

Remark 1: <u>h</u> = 0, Kenyon-Mossel-Peres 01

$$\tau_{\rm mix}(G) \le \exp\{2\beta\Gamma(G;0)\}.$$

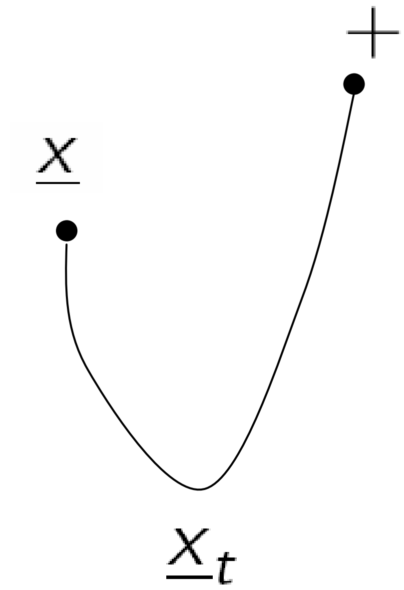
Remark 2: at ¯ = 1 recovers Morris 00

$$\max_{F\subseteq G}\Gamma(F;\underline{h}^F)=0$$

Proof idea

Hitting time is related to the size of barrier: Variation of Markov chain comparison (Diaconis-Saloff coste, Jerrum-Sinclair)

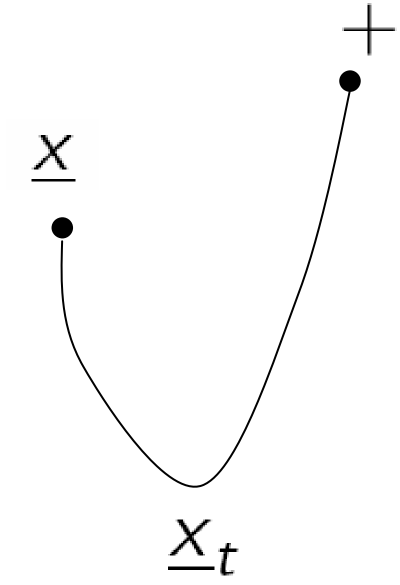
$$\Gamma_*(G;\underline{h}) \simeq \max_{\underline{x}} \min_{\text{path}} \max_{t} H(\underline{x}) - H(\underline{x}_t)$$



Proof idea

Lemma: There is a path with minimum barrier that is monotone (flips only – to +)

$$\min_{\text{monotone path}} \max_t H(\underline{x}) - H(\underline{x}_t)$$

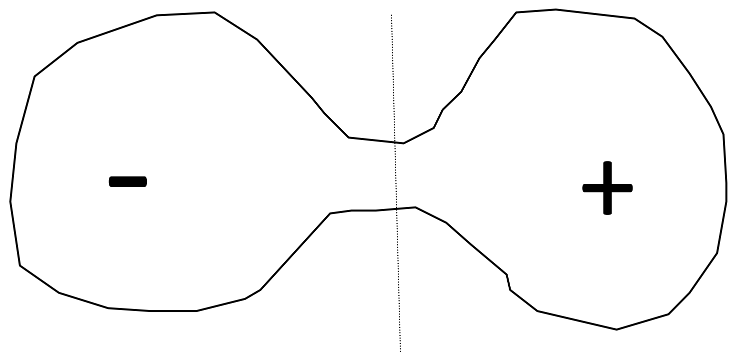


Tilted cutwidth is the ordering defined by minimum path defined by $\underline{x}$

Also a dual notion

Worst Bottleneck: gives a lower bound

$$\Delta(G;\underline{h}) \equiv \max_{\Omega} \min_{(S_1,S_2)\in\partial\Omega} \max_{i=1,2} [\mathrm{cut}(S_i, V \setminus S_i) - |S_i|_h]$$



Tilted cut is equal to the tilted cutwidth in the worst subgraph!

Our goal

- Find the graph theoretic property that captures convergence

- Demonstrate on specific graph sequences
  - Expanders
  - Finite range graphs
  - Small-worlds

Expanders

G is (λ, δ)-expander if and only if

$$\forall S \subset V, |S| \leq \delta n, \ \mathrm{cut}(S, V \setminus S) \geq \lambda |S|$$

If G is a (λ, δ)-expander then $\Gamma_*(G; \underline{h}) \geq (\lambda - h_{\max})\delta n$.

Proof:

$$\begin{aligned} \Gamma_*(G; \underline{h}) \geq \Delta(G, \underline{h}) &\geq \min_{|S|=t} [\mathrm{cut}(S, V \setminus S) - |S|_h] \\ &\geq \min_{|S|=t} [\lambda |S| - h_{\max} |S|] \end{aligned}$$

Expanders

G is (λ, δ)-expander if and only if

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If G is a (λ, δ)-expander then $\Gamma_*(G; \underline{h}) \geq (\lambda - h_{\max})\delta n$.

Corollary: Convergence time is exponential for random regular and power-law graphs! (different from Ellison's prediction)

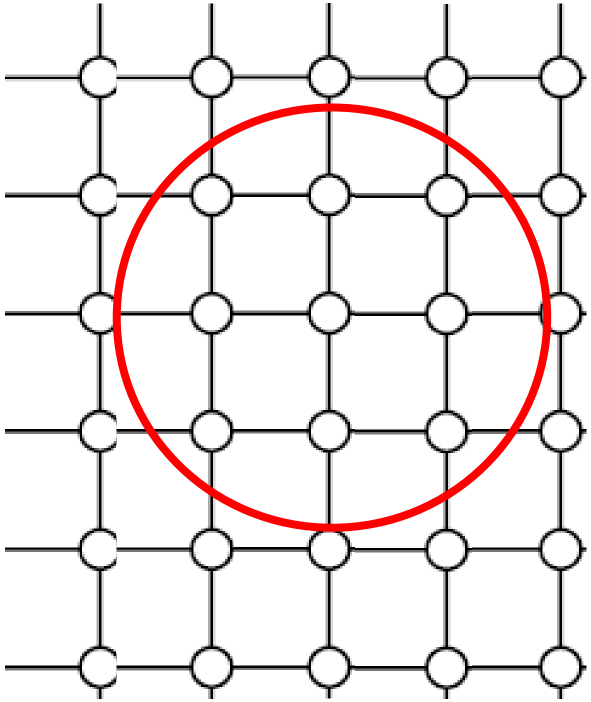
Convergence and Isoperimetric function

If for some $\alpha < 1$, and $\forall V \subseteq V(G)$

$\exists S \subset V, \mathrm{cut}(S, V \setminus S) \leq |S|^{\alpha}$

Then

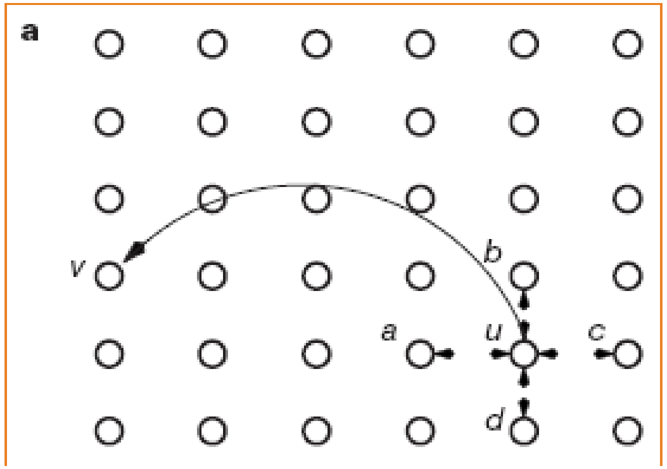
$\Gamma_*(G; \underline{h}) = O(1)$



Corollary: Convergence time is fast for all low-dimensional graphs (e.g. k-dimensional grids)

Small-world networks

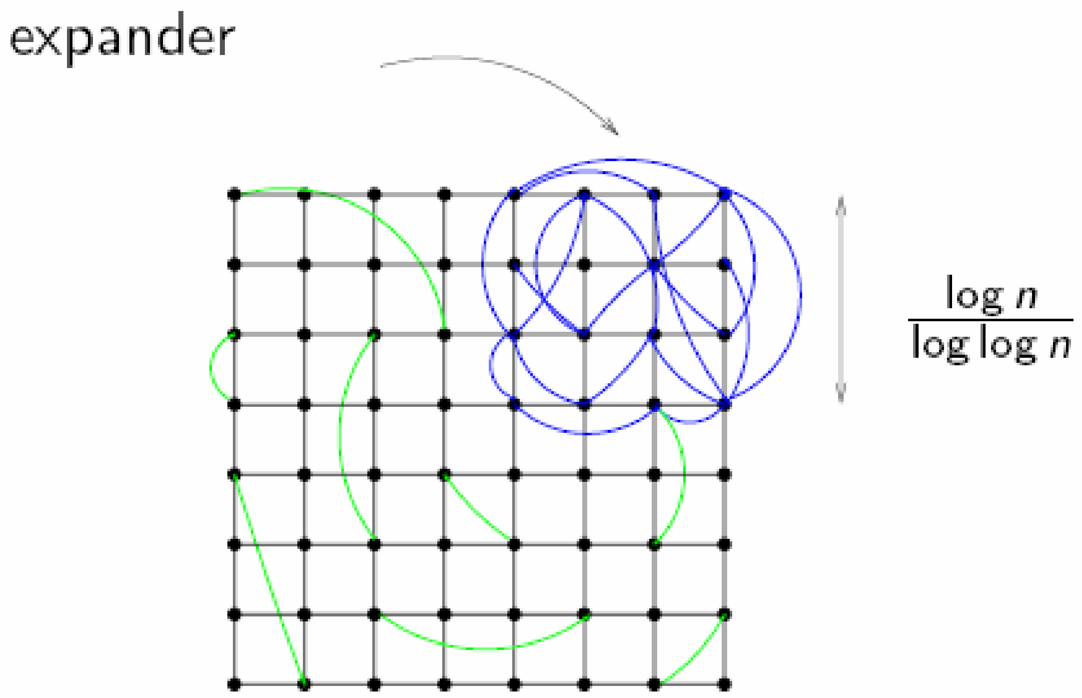
- Watts-Strogatz, Kleinberg 95



$$P(u, v) \propto \frac{1}{d(u, v)^r}$$

Small-world networks

If G is small world with $r \geq d$, then $\Gamma_*(G; \underline{h}) = \Omega\left(\frac{\log n}{\log \log n}\right)$.



Small world graphs with $r < d$ have constant expansion!

Summary

Consider asynchronous reversible dynamics.

If G is:

d-dimensional finite-range,	then $\tau_+(G) = e^{\beta\Theta(d)}$.
has constant expansion,	then $\tau_+(G) = e^{\beta\Omega(n)}$.
small world with $r > d$,	then $\tau_+(G) = e^{\beta\Omega(\log n / \log\log n)}$.
small world with $r \leq d$,	then $\tau_+(G) = e^{\beta\Omega(n)}$.

Comparison with epidemic models

Epidemic models: the behavior spreads from an infected node to a healthy neighbor independently

e.g. the contact process

In contrast with the dynamics of coordination games

High degree nodes and well-connected subgraphs help the spread of behavior in contagion dynamics
(e.g. Berger, Borgs, Chayes, S. 04)

Which model is more realistic?

Case study:

- MusicLab (Salagnik-Watts)
  - 14k participants
  - 8 independent worlds

- Communities in live journal:
  - 1 million users
  - 2200 communities

- ``happiness'' in blogsphere:
  - Crawling blogger, myspace etc..
  - 15,000 new feelings per day!

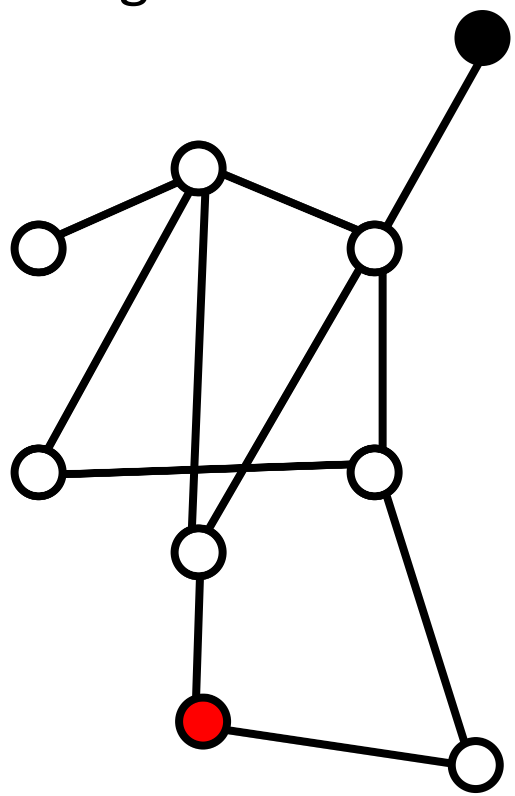
Joint work with S. Ceyhan and S. Kamvar

Dynamics with heterogeneous users

- Zealots: a small set of individual who don't change their behavior

Typical behavior?

- Oscillation between the alternatives
- Well-defined territories



Centrality of zealots, connectivity of the graph

work in progress with D. Acemoglu and A. Ozdaglar

Summary & open problems

Coordination games

- Rate of convergence and network structure
- "Small number of connections" is not enough for fast convergence
- More precise measure: tilted cutwidth or isoperimetric function of the graph

To do

- Obtain a similar understanding of other games (especially non-potential games)
- Algorithms for computing tilted cutwidth/predicting the outcome of dynamics