Probabilistic Cat-and-Mouse Game

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Edinburgh June 8 2010

Outline

- On-line algorithm for solving Markov chains
- The Cat and Mouse process
- Stationary distribution
- Scaling results

On-line algorithm for solving Markov chains

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  - Advanced linear algebra methods to speed up power iterations (off-line)
  - Monte Carlo methods (off-line or on-line)
  - Other non-trivial on-line methods. One such method by Abiteboul, Preda and Cobena (1999)

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- The estimator of π_i at time t is $\pi_n(x) = \dfrac{h_n(x)}{\sum_y h(y).}$
- $\pi_n(x) \to \pi(x)$ a.s. as $n \to \infty$.

The cat and mouse process

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- At time n, the mouse is at the node M_n.
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- We want to analyse (C_n, M_n)

Relation to the cash process

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Theorem. For $t \geq 0$,

$$(V_n(x), x \in \mathcal{S}) \overset{\text{dist.}}{=} (\mathbb{P}[M_n = x \mid \mathcal{F}_{n-1}], x \in \mathcal{S}) \,.$$

In particular, $\mathbb{E}(V_n(x)) = \mathbb{P}(M_n = x)$.

Proof for the expectation

C_n – cat position; M_n – mouse position; $V_n(x)$ – cash at node x.

Show that $\mathbb{E}(V_n(x)) = \mathbb{P}(M_n = x)$.

$$\mathbb{P}(M_{n+1} = x \mid \mathcal{F}_n) = \sum_{y \neq x} 1_{\{C_n = y\}} p(y,x) \mathbb{P}(M_n = y \mid \mathcal{F}_{n-1}) + 1_{\{C_n \neq x\}} \mathbb{P}(M_n = x \mid \mathcal{F}_{n-1}).$$

On the other hand, for the cash

$$\mathbb{E}[V_{n+1}(x)] = \sum_{y \neq x} 1_{\{C_n = y\}} p(y,x) \mathbb{E}[V_n(x)] + 1_{\{C_n \neq x\}} \mathbb{E}[V_n(x)].$$

Same equation! Statement follows by induction in n.

Asymptotic behavior

Stationary distribution of the Cat and Mouse chain

- there exists some constant c such that,
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Proposition. The stationary distribution of the mouse is

$$\mathbb{P}(M_\infty = x) = c\mathbb{E}_\pi \left[p(C_0, x)\, T_x\right]$$

and

$$c = \left[\sum_z \mathbb{E}_\pi \left[p(C_0, z)\, T_z\right]\right]^{-1}$$

Reversible Markov chain

If (C_n) is reversible then we get

$$c = \frac{1}{N-1} \text{ and } \mathbb{P}(M_\infty = y) = \frac{1-\pi(y)}{N-1}.$$

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The mouse is less likely to find in more popular states. This is however not true in general.

Scaling results

Symmetric random walk on $\mathbb{Z}$

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- Then $M_n \sim \sqrt[4]{n}$

Scaling result for the symmetric random walk on $\mathbb{Z}$

- **Theorem.** $(B_1(t))$, $(B_2(t))$ – two independent Brownian motions, $L_{B_2}(t)$ is the local time of $(B_2(t))$. Then

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt[4]{n}}M_{\lfloor nt\rfloor},\ t\geq 0\right)\stackrel{dist}{\rightarrow}(B_1(L_{B_2}(t)))$$

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- **Intuition:**
  - Series of joint steps of the cat and the mouse: on average, 2 steps per series
  - Between the series: approximately twice a renewal cycle of the symmetric random walk.

Reflected random walk: $M/M/1$ case

- The cat follows a simple ergodic random walk on $\mathbb{Z}_+$ with reflection at 0.
- $p(x, x+1) = p < 1/2$; $p(x, x-1) = q = 1 - p$, $x \neq 0$; $p(0, 0) = q$.

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- Let $T_n = \inf\{k > 0 : C_k = n\}$.
- It is known that $T_n/\mathbb{E}_0(T_n)$ converges to an exponentially distr. r.v. as $n \to \infty$ and

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- Correct time scale: $t \to t\rho^{-n}$

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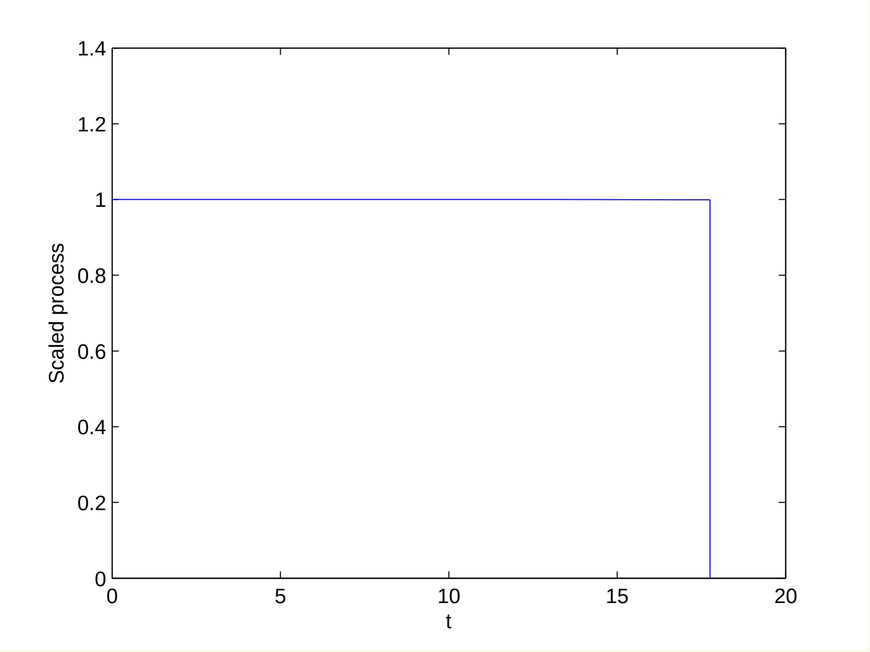

Return time to zero of the mouse

- **Theorem.** If $(C_0, M_0) = (0, n)$ and $H_0 = \inf\{k > 0 : M_k = 0\}$ then

$$\lim_{n\to\infty} \rho^{-n} H_0 \stackrel{dist}{=} W = \sum_{k=0}^{\infty} \rho^{-S_k} E_k$$

- S_k – random walk associated with the free process of the mouse
- $E_1, E_2, \ldots$ – independent exponential r.v.'s with mean 1.

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- W is extremely heavy-tailed: $\mathbb{E}[W] = \infty$

Scaled behaviour of the mouse

- **Theorem.**

$$\lim_{n\to\infty}\left(\frac{M_{\lfloor t\rho^{-n}\rfloor}}{n}\mathbf{1}_{\{0\le t\rho^{-n}\le H_0\}}\right)\overset{dist}{=}\left(\mathbf{1}_{\{t<W\}}\right)$$

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- Initially we thought the scaled process stays at zero... but a simple counting argument tells us that this is not true!

What happens after W?

- Assume that the cat is at zero and the mouse at δn, $\delta < 1$. In time $\rho^{-n} t$ the cat comes to δn on average

$$c\rho^{\delta n}\rho^{-n}t = c\rho^{-n(1-\delta)}t$$

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- **Theorem.** If the cat and the mouse start at zero then the scaled process $\frac{M_{\lfloor \{\rho^{-n}\}t \rfloor}}{n}$ visits the level $\delta < 1/2$ infinitely many times on $[0, t)$.

Other scenarios

- Symmetric random walk in Z^2: $C_n \sim \sqrt{n}$, $M_n \sim \sqrt{\log n}$

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- Symmetric random walk in Z^2: $C_n \sim \sqrt{n}$, $M_n \sim \sqrt{\log n}$
- $M/M/\infty$ random walk in cont. time: if $(C(0), M(0)) = (n, n)$ then $M(T_0) \sim Fn$, where $P(F \leq x) = x^\rho$, $x \in [0, 1]$.

Technical tools

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- Limit Theorems for additive functionals of Markov processes: Knight (1962) Borodin (1981), Perkins (1982), Kasahara (1982, 1985)
- Coupling arguments, technical estimates, scaling results for Markov chains

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- The End