Stochastic Population Dynamics and
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Stochastic dynamics of infinite particle systems

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Birth-and-death dynamics

$$\begin{aligned}(LF)(\gamma) &= \sum_{x\in\gamma} d(x,\gamma\setminus x)\big(F(\gamma\setminus x)-F(\gamma)\big)\\ &+ \int_{\mathbb{R}^d} b(x,\gamma)\big(F(\gamma\cup x)-F(\gamma)\big)\,dx\end{aligned}$$

Contact model:

$$d\equiv 1,\quad b(x,\gamma)=\lambda\sum_{y\in\gamma}a(x-y)$$

Voter model:

$$d(x,\gamma)=\sum_{y\in\gamma}a_-(x-y),\quad b(x,\gamma)=\sum_{y\in\gamma}a_+(x-y)$$

Glauber dynamics:

$$d\equiv 1,\quad b(x,\gamma)=e^{-E(x,\gamma)}\left(E(x,\gamma):=\sum_{y\in\gamma}\phi(x-y)\right)$$

We consider birth and death rates of the type

$$b(x,\gamma)=(KB_x)(\gamma)\ge 0,\quad d(x,\gamma)=(KD_x)(\gamma)\ge 0$$

Hopping particle systems

$$(LF)(\gamma) = \sum_{x\in\gamma} \int_{\mathbb{R}^d} c(x, y, \gamma)\,(F(\gamma \setminus x \cup y) - F(\gamma))\; dy$$

Kawasaki dynamics:

$$c(x, y, \gamma) = a(x - y)e^{-E(y,\gamma)}$$

We consider the special case

$$c(x, y, \gamma) = (KC_{x,y})(\gamma \setminus x) \geq 0$$

- **The configuration space Γ**

$$\Gamma := \{\gamma \subset \mathbb{R}^d : |\gamma \cap K| < \infty, \forall \text{ compact } K \subset \mathbb{R}^d\}$$

Each $\gamma \in \Gamma$ is identified with a Radon measure:

$$\Gamma \ni \gamma \mapsto \sum_{x \in \gamma} \delta_x \in \mathcal{D}'(\mathbb{R}^d) \quad \text{(configuration)}$$

$\delta_x :=$ the Dirac measure with mass at x

Vague topology:

$$\Gamma \ni \gamma \longmapsto \sum_{x \in \gamma} f(x) \text{ continuous}$$

f: continuous function with compact support

- **The finite configuration space Γ_0**

$$\Gamma_0 := \bigsqcup_{n=0}^{\infty} \underbrace{\{\gamma : |\gamma| = n\}}_{:=\Gamma^{(n)}}$$

Combinatorial harmonic analysis

Given a $G : \Gamma_0 \to \mathbb{C}$ one associates $KG : \Gamma \to \mathbb{C}$

$$(KG)(\gamma) := \sum_{\substack{\eta \subset \gamma \\ |\eta| < \infty}} G(\eta)$$

Coherent states:

$$\Big(K \underbrace{\prod_{x \in \cdot} \theta(x)}_{e_\lambda(\theta)}\Big)(\gamma) = \prod_{x \in \gamma} (1 + \theta(x))$$

- $K e_\lambda(0) \equiv 1$

(death rate of contact and Glauber dynamics)

- $K e_\lambda(e^{-\phi(x - \cdot)} - 1)(\gamma) = e^{-E(x,\gamma)}$

(birth rate of Glauber dynamics)

Bounded functions with bounded support:

$$|G| \le C 1\!\!1_{\bigsqcup_{n=0}^{N} \Gamma_\Lambda^{(n)}} \quad (G \in B_{bs}(\Gamma_0))$$

- $|(KG)(\gamma)| = |(KG)(\gamma_\Lambda)| \le C(1 + |\gamma_\Lambda|)^N$

(where $\gamma_\Lambda := \gamma \cap \Lambda, \Gamma_\Lambda^{(n)} := \{\eta \in \Gamma^{(n)} : \eta \subset \Lambda\}$)

- $K : B_{bs}(\Gamma_0) \to K(B_{bs}(\Gamma_0))$ linear isomorphism:

$$\left(K^{-1} F\right)(\eta) := \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} F(\xi), \quad \eta \in \Gamma_0$$

Correlation measures

μ probability measure on Γ s.t.

$$\int_\Gamma |\gamma \cap \Lambda|^n \, d\mu(\gamma) < \infty, \quad n \in \mathbb{N}_0, \Lambda \in \mathcal{B}_c(\mathbb{R}^d)$$

Correlation measure ρ_μ corresponding to μ:
Measure defined on Γ_0 by

$$\int_{\Gamma_0} G(\eta)\, d\rho_\mu(\eta) = \int_\Gamma (KG)(\gamma)\, d\mu(\gamma)$$

for all $G \in B_{\mathrm{bs}}(\Gamma_0)$

- $B_{\mathrm{bs}}(\Gamma_0) \subset L^1(\Gamma_0, \rho_\mu)$. Moreover,

$$\|KG\|_{L^1(\mu)} \leq \|G\|_{L^1(\rho_\mu)} \Longrightarrow K : L^1(\Gamma_0, \rho_\mu) \to L^1(\Gamma, \mu)$$

bounded linear operator

Example: Poisson measure π_σ

$$\int_\Gamma \exp\Big(\sum_{x\in\gamma} \varphi(x)\Big)\, \pi_\sigma(d\gamma) = \exp\Big(\int_{\mathbb{R}^d} (e^{\varphi(x)} - 1)\, \sigma(dx)\Big)$$

Then,

$$\rho_{\pi_\sigma} := \lambda_\sigma = \sum_{n=0}^\infty \frac{1}{n!} \sigma^{(n)}$$

(Lebesgue-Poisson measure)

$$\frac{\partial}{\partial t}F_t = LF_t \qquad \langle F, \mu \rangle = \int_\Gamma d\mu(\gamma)F(\gamma) \qquad \frac{d}{dt}\mu_t = L^*\mu_t$$

$$\begin{array}{ccc} F & \longleftrightarrow & \mu \\ {\scriptstyle K}\uparrow & & \downarrow{\scriptstyle K^*} \\ G & \longleftrightarrow & \rho_\mu \end{array}$$

$$\frac{\partial}{\partial t}G_t = \widehat{L}G_t \qquad \langle G, \rho_\mu \rangle = \int_{\Gamma_0} d\rho_\mu(\eta)G(\eta) \qquad \frac{\partial}{\partial t}k_t = \widehat{L}^*k_t$$

$$\frac{d}{dt}\int_{\Gamma_0} G \underbrace{k_t\, d\lambda}_{d\rho_t} = \frac{d}{dt}\int_\Gamma KG\, d\mu_t = \int_\Gamma L(KG)\, d\mu_t = \int_{\Gamma_0} ((\underbrace{K^{-1}LK}_{\widehat{L}})G)k_t\, d\lambda$$

Birth-and-death dynamics

$$(LF)(\gamma) = \sum_{x\in\gamma} d(x,\gamma\setminus x)\big(F(\gamma\setminus x)-F(\gamma)\big) + \int_{\mathbb{R}^d} b(x,\gamma)\big(F(\gamma\cup x)-F(\gamma)\big)\,dx$$

with

$$b(x,\gamma)=(KB_x)(\gamma)\geq 0,\quad d(x,\gamma)=(KD_x)(\gamma)\geq 0$$

$$(KG_1)\cdot(KG_2)=K(G_1\star G_2)$$

where

$$(G_1\star G_2)(\eta):=\sum_{\eta_1\cup\eta_2=\eta} G_1(\eta_1)G_2(\eta_2)$$

For $F=KG\in K(B_{bs}(\Gamma_0))$ we find

$$(LF)(\gamma) = -\sum_{x\in\gamma}\left(K\left(D_x\star G(\cdot\cup x)\right)\right)(\gamma\setminus x) + \int_{\{x:x\notin\gamma\}}\left(K\left(B_x\star G(\cdot\cup x)\right)\right)(\gamma)\,dx,$$

which yields an explicit formula for $\hat{L}:=K^{-1}LK$:

$$(\hat{L}G)(\eta) = -\sum_{x\in\eta}\left(D_x\star G(\cdot\cup x)\right)(\eta\setminus x) + \int_{\mathbb{R}^d}\left(B_x\star G(\cdot\cup x)\right)(\eta)\,dx$$

$$(\widehat{L}G)(\eta) = -\sum_{x\in\eta} \left(D_x \star G(\cdot\cup x)\right)(\eta\backslash x) + \int_{\mathbb{R}^d} \left(B_x \star G(\cdot \cup x)\right)(\eta)\, dx$$

In terms of correlation functions:

$$\int_{\Gamma_0} d\lambda(\eta)\, (\widehat{L}^* k)(\eta) G(\eta) = \int_{\Gamma_0} d\lambda(\eta)\, (\widehat{L}G)(\eta) k(\eta)$$

This means that for λ-a.a. $\eta \in \Gamma_0$

$$\begin{aligned}(\widehat{L}^* k)(\eta) &= -\int_{\Gamma_0} d\lambda(\zeta)\, k(\zeta\cup\eta) \sum_{x\in\eta}\sum_{\xi\subset\eta\backslash x} D_x(\zeta\cup\xi) \\ &\quad + \int_{\Gamma_0} d\lambda(\zeta) \sum_{x\in\eta} k(\zeta\cup(\eta\setminus x)) \sum_{\xi\subset\eta\backslash x} B_x(\zeta\cup\xi)\end{aligned}$$

In terms of generating functionals: $(G = e_\lambda(\theta))$

$$\begin{aligned}\frac{\partial}{\partial t} \underbrace{\int_{\Gamma_0} d\lambda(\eta) e_\lambda(\theta,\eta) k_t(\eta)}_{:=B_t(\theta)} &= \int_{\Gamma_0} d\lambda(\eta)\, (\widehat{L}^* k_t)(\eta) e_\lambda(\theta,\eta) \\ &= \underbrace{\int_{\Gamma_0} d\lambda(\eta) (\widehat{L} e_\lambda(\theta))(\eta) k_t(\eta)}_{??=\tilde{L}B_t??},\end{aligned}$$

where

$$B_t(\theta) = \int_{\Gamma_0} e_\lambda(\theta,\eta) k_t d\lambda(\eta) = \int_{\Gamma} \prod_{x\in\gamma} (1+\theta(x)) \mu_t(d\gamma)$$

(**Bogoliubov functional**)

Under analytical assumptions,

$$\begin{aligned}&(\tilde{L}B_t)(\theta) = \\ &-\int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(\theta+1\,\eta) \int_{\mathbb{R}^d} dx\, \theta(x) (D^{|\eta|+1} B_t)(\theta, \eta\cup x) D_x(\eta) \\ &+ \int_{\Gamma_0} d\lambda(\eta)\, (D^{|\eta|} B_t)(\theta,\eta) e_\lambda(\theta+1,\eta) \int_{\mathbb{R}^d} dx\, \theta(x) B_x(\eta)\end{aligned}$$

The Glauber dynamics:

$$(LF)(\gamma) = \sum_{x\in\gamma} \big(F(\gamma\setminus x) - F(\gamma)\big) + \int_{\mathbb{R}^d} e^{-E(x,\gamma)}\big(F(\gamma\cup x) - F(\gamma)\big)\,dx$$

Generators:

- $$(\widehat{L}G)(\eta) = -|\eta|G(\eta) + \int_{\mathbb{R}^d} \Big(e_\lambda(e^{-\phi(x-\cdot)} - 1) \star G(\cdot \cup \{x\})\Big)(\eta)\,dx,$$

- $$(\widehat{L}^*k)(\eta) = -|\eta|k(\eta) + \sum_{x\in\eta} e^{-E(x,\eta\setminus x)} \int_{\Gamma_0} e_\lambda(e^{-\phi(x-\cdot)} - 1, \zeta) k((\eta\setminus x)\cup\zeta)\,d\lambda(\zeta)$$

- $$-(\widetilde{L}B)(\theta) = \int_{\mathbb{R}^d} dx\,\theta(x)\left(\frac{\delta B(\theta)}{\delta\theta(x)} - B\Big((1+\theta)\big(e^{-\phi(x-\cdot)} - 1\big) + \theta\Big)\right)$$

Remark:
μ Gibbs measure $\Longleftrightarrow B_\mu$ solution of the Bogoliubov equation

$$\frac{\delta B(\theta)}{\delta\theta(x)} = B\left((1+\theta)\left(e^{-\beta\phi(x,\cdot)} - 1\right) + \theta\right)$$

for all $\theta \in L^1(dx), dx$-a.a. $x \in \mathbb{R}^d$.

The contact model:

$$\begin{aligned}(LF)(\gamma) &= \sum_{x\in\gamma}\big(F(\gamma\setminus x)-F(\gamma)\big)\\ &+\lambda\int_{\mathbb{R}^d}\sum_{y\in\gamma}a(x-y)\big(F(\gamma\cup x)-F(\gamma)\big)\,dx\end{aligned}$$

Generators:

- $$(\widehat{L}G)(\eta) = -|\eta|G(\eta)$$ $$+\lambda\sum_{y\in\eta}\int_{\mathbb{R}^d}a(x-y)\,(G(\eta\cup x)+G((\eta\setminus y)\cup x))\;dx$$

- $$(\widehat{L}^*k)(\eta) = -|\eta|k(\eta)$$ $$+\lambda\int_{\mathbb{R}^d}dy\,\sum_{x\in\eta}k((\eta\setminus x)\cup y)a(x-y)$$ $$+\lambda\sum_{x\in\eta}k(\eta\setminus x)\sum_{y\in\eta\setminus x}a(x-y)$$

- $$(\tilde{L}B)(\theta) = -\int_{\mathbb{R}^d}dx\,\theta(x)\frac{\delta B(\theta)}{\delta\theta(x)}$$ $$+\lambda\int_{\mathbb{R}^d}dy\int_{\mathbb{R}^d}dx\,a(x-y)(1+\theta(y))\theta(x)\frac{\delta B(\theta)}{\delta\theta(y)}$$

The voter model:

$$
\begin{aligned}
(LF)(\gamma) &= \sum_{x\in\gamma}\sum_{y\in\gamma\setminus x} a_-(x-y)\big(F(\gamma\setminus x)-F(\gamma)\big)\\
&\quad+\int_{\mathbb{R}^d}\sum_{y\in\gamma} a_+(x-y)\big(F(\gamma\cup x)-F(\gamma)\big)\,dx
\end{aligned}
$$

Generators:

- $$(\hat{L}G)(\eta) = -\sum_{x\in\eta}\sum_{y\in\eta\setminus x} a_-(x-y)\,(G(\eta\setminus y)+G(\eta))$$
  $$+\sum_{y\in\eta}\int_{\mathbb{R}^d} a_+(x-y)\,(G(\eta\cup x)+G((\eta\setminus y)\cup x))\;dx$$

- $$(\hat{L}^*k)(\eta) = -\int_{\mathbb{R}^d} dy\,k(\eta\cup y)\sum_{x\in\eta} a_-(x-y)$$
  $$-k(\eta)\sum_{x\in\eta}\sum_{y\in\eta\setminus x} a_-(x-y)$$
  $$+\int_{\mathbb{R}^d} dy\sum_{x\in\eta} k((\eta\setminus x)\cup y)a_+(x-y)$$
  $$+\sum_{x\in\eta} k_t(\eta\setminus x)\sum_{y\in\eta\setminus x} a_+(x-y)$$

- $$(\tilde{L}B)(\theta) = \int_{\mathbb{R}^d} dy\int_{\mathbb{R}^d} dx\,a_+(x-y)(1+\theta(y))\theta(x)\frac{\delta B(\theta)}{\delta\theta(y)}$$
  $$-\int_{\mathbb{R}^d} dy\int_{\mathbb{R}^d} dx\,a_-(x-y)(1+\theta(y))\theta(x)\frac{\delta^2 B(\theta)}{\delta\theta(x)\delta\theta(y)}$$

Hopping particles: the general case

$$(LF)(\gamma) = \sum_{x\in\gamma} \int_{\mathbb{R}^d} c(x,y,\gamma)\,(F(\gamma\setminus x\cup y) - F(\gamma))\,dy$$

with

$$c(x,y,\gamma) = (KC_{x,y})(\gamma\setminus x) \geq 0$$

Similar techniques yield

$$(\hat{L}G)(\eta) = \sum_{x\in\eta} \int_{\mathbb{R}^d} (C_{x,y} \star (G(\cdot\cup y) - G(\cdot\cup x)))\,(\eta\setminus x)\,dy$$

In terms of correlation functions:

$$\begin{aligned}(\hat{L}^*k)(\eta) &= \sum_{y\in\eta} \int_{\mathbb{R}^d} dx \int_{\Gamma_0} d\lambda(\xi)\, k(\xi\cup\eta\setminus y\cup x) \sum_{\zeta\subset\eta\setminus y} C_{x,y}(\xi\cup\zeta) \\ &\quad - \int_{\Gamma_0} d\lambda(\xi)\, k(\xi\cup\eta) \sum_{x\in\eta} \sum_{\zeta\subset\eta\setminus x} \int_{\mathbb{R}^d} dy\, C_{x,y}(\xi\cup\zeta)\end{aligned}$$

In terms of generating Bogoliubov functionals:

$$\begin{aligned}(\tilde{L}B)(\theta) &= \int_{\Gamma_0} d\lambda(\eta) e_\lambda(\theta+1,\eta) \int_{\mathbb{R}^d} dx\, (D^{|\eta|+1}B)(\theta,\eta\cup x) \\ &\quad \cdot \int_{\mathbb{R}^d} dy\, (\theta(y) - \theta(x)) C_{x,y}(\eta)\end{aligned}$$

Remark: In the kawasaki case $\left(c(x,y,\gamma) = a(x-y)e^{-E(y,\gamma)}\right)$

$$\begin{aligned}(\tilde{L}B)(\theta) &= \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, a(x-y) e^{-\phi(x-y)} (\theta(y) - \theta(x)) \\ &\quad \times \frac{\delta B((1+\theta)(e^{-\phi(y-\cdot)} - 1) + \theta)}{\delta((1+\theta)(e^{-\phi(y-\cdot)} - 1) + \theta)(x)}\end{aligned}$$