Scaling limit for jump, birth and death type dynamics

Tobias Kuna

University of Reading

Workshop on Stochastic Population Dynamics and
Application in Spatial Ecology 2009
19.6.2009, Edinburgh

.

in collaboration with
T. Funaki, D. Finkelshtein, Yu. Kondratiev, O. Kutoviy,

Model and Motivation

- particles in the continuum $\mathbb{R}^n$.

$$\gamma(dx) = "\gamma(x)d^n x" = \sum_{i \in I} \delta_{x_i}(x) d^n x$$

Bolker: $N(x) = \gamma(x)d^n x$

- potentially infinite many particles
- Spatial point process $\mathbb{P}$
- Expectation: $\langle F(\gamma) \rangle_{\mathbb{P}} = \int F(\gamma) dP(\gamma)$
- Expected density at x

$$\langle g(d^n x) \rangle = \rho(x) d^n x$$

- Weak form: f nice function on $\mathbb{R}^n$.

$$\int_{\mathbb{R}^n} f(x)\gamma(dx) = \sum_{i\in I}\int_{\mathbb{R}^n} f(x)\delta_{x_i}(x)d^n x = \sum_{i=1}^{n} f(x_i)$$

- $f(x) = 1$ on $\Delta \subset \mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} f(x)\gamma(dx) = \gamma(\Delta) = \#\{i \in I : x_i \in \Delta\} = N_\Delta(\gamma)$$

- Discretization: $\mathbb{R}^n = \cup_{j\in\mathbb{Z}^n}\Delta_j$ and $f(x) = a_j$ if $x \in \Delta_j$. Then

$$\int_{\mathbb{R}^n} f(x)\gamma(dx) = \sum_j f_j \,\#\{i \in I : x_i \in \Delta_j\} = \sum_j f_j \gamma(\Delta_j) = \sum_j f_j N(j)$$

where $N(j) = \#\{i \in I : x_i \in \Delta_j\} = \gamma(\Delta_j)$.

Birth generator

- Birth

$$N(j) \to N(j) + \delta_i(j) = \begin{cases} N(j) & \text{for } j \neq i \\ N(j) + 1 & \text{for } j = i \end{cases}, \qquad \text{rate } B(i, N(\cdot))$$

in the continuum

$$\gamma(d^n x) \to \gamma(d^n x) + \delta_y(dx^n)$$

Rate: birth from site j into i

$$N(j)B(j, i, N(\cdot))$$

Generator

$$(L_b F)(N(\cdot)) = \sum_i \sum_j N(j)B(i, j, N(\cdot)) \left(F(N(\cdot) + \delta_i(\cdot)) - F(N(\cdot))\right)$$

Birth, death generator

- in the continuum

$$(L_b F)(\gamma) \quad = \quad \int_{\mathbb{R}^n} d^n x \int_{\mathbb{R}^n} \gamma(d^n y) B(x, y, \gamma) \bigl(F(\gamma + \delta_y) - F(\gamma)$$

- Independent birth: $B(y, \gamma) = B_0$.
- Death

$$(L_d F)(\gamma) \quad = \quad \int_{\mathbb{R}^n} \gamma(d^n y)\, D(y, \gamma) \bigl(F(\gamma - \delta_y) - F(\gamma)\bigr)\,.$$

- Independent death $D(y, \gamma) = D_0$.

Jump generator, dynamics

- Jump

$$(L_jF)(\gamma) = \int_{\mathbb{R}^n} d^n y \int_{\mathbb{R}^n} \gamma(d^n x) A(x,y,\gamma) \big(F(\gamma + \delta_y - \delta_x) - F(\gamma)\big)$$

- Dynamics

$$\frac{d}{dt}F(\gamma_t) = LF(\gamma_t) + \frac{d}{dt}M(F,t)$$

$M(F,t)$ martingale, in particular $\langle M(F,t)\rangle = 0$.

- Case $F(\gamma) = \int_{\mathbb{R}^n} f(x)\gamma(dx)$, then

$$\begin{aligned}
(L_bF)(\gamma) &= \int_{\mathbb{R}^n} d^n y \int_{\mathbb{R}^n} \gamma(d^n x) B(x,y,\gamma) f(y) \\
(L_dF)(\gamma) &= -\int_{\mathbb{R}^n} \gamma(d^n x) d(x,\gamma) f(x) \\
(L_jF)(\gamma) &= \int_{\mathbb{R}^n} dy \int_{\mathbb{R}^n} \gamma(d^n x) A(x,y,\gamma)(f(y) - f(x))
\end{aligned}$$

Dynamic of first correlation function

- Definition $\rho_t(x)d^nx = \langle \gamma_t(d^nx) \rangle$ Hence

$$\int_{\mathbb{R}^n} d^nx\, f(x)\rho_t(x) = \langle \int_{\mathbb{R}^n} f(x)\gamma_t(d^nx) \rangle$$

Thus

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^n} d^nx\, f(x)\rho_t(x) = \int_{\mathbb{R}^n} d^ny \int_{\mathbb{R}^n} \langle \gamma(d^nx) B(x,y,\gamma) \rangle f(y)\\ &- \int_{\mathbb{R}^n} \langle \gamma(d^nx) d(x,\gamma) \rangle f(x)\\ &+ \int_{\mathbb{R}^n} d^ny \int_{\mathbb{R}^n} \langle \gamma(d^nx) A(x,y,\gamma) \rangle (f(y) - f(x))\\ &+ \frac{d}{dt}\langle M(f,\gamma_t) \rangle\end{aligned}$$

If for example $d(x,\gamma) = \int_{\mathbb{R}^n} r(x-y)\gamma(d^ny)$, then

$$\langle \gamma(d^nx) d(x,\gamma) \rangle = \int_{\mathbb{R}^n} r(x-y) \langle \gamma(d^ny)\gamma(d^nx) \rangle$$

where $\langle \gamma(d^ny)\gamma(d^nx) \rangle = 2c(x,y)d^nxd^ny + \delta(x-y)\rho(x)d^nx$

Space scaling

- Discretization: $\mathbb{R}^n = \cup_{j \in \mathbb{Z}^n} L\Delta_j$ and $f_L(x) = a_j$ if $x \in \Delta_j$. Then

$$\int_{\mathbb{R}^n} f_L(x)\gamma(dx) = \sum_j a_j \#\{i \in I : x_i \in L\Delta_j\}$$

Grows in L often like L^d. Normalize

$$\begin{aligned} & \frac{1}{L^d} \sum_j a_j \#\{i \in I : x_i \in L\Delta_j\} \\ = \quad & \frac{1}{L^d} \sum_j a_j \#\{i \in I : x_i/L \in \Delta_j\} = \int_{\mathbb{R}^n} f(x)\gamma_L(d^n x) \end{aligned}$$

where

$$\gamma_L(d^n x) = L^{-d} \sum_i \delta_{x_i/L}(d^n x)$$

- Then

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^n} f(x)\rho_L(t,dx) &= \frac{d}{dt}\langle\int_{\mathbb{R}^n} f(x)\gamma_L(t,dx)\rangle && (1)\\ &= L^{-d}\frac{d}{dt}\int_{\mathbb{R}^n} f(x/L)\gamma(dx) && (2)\end{aligned}$$

- Jump part

$$\begin{aligned} & L^{-d}\int_{\mathbb{R}^n} d^n y\int_{\mathbb{R}^n}\gamma(d^n x)A(x,y,\gamma)\left(f(y/L)-f(x)\right)\\ &= \int_{\mathbb{R}^n} d^n y\int_{\mathbb{R}^n}\gamma_L(d^n x)A(xL,y,\gamma)\left(f(y/L)-f(x)\right)\end{aligned}$$

- Translation invariance

$$A(x,y,\gamma) = A(0,y-x,\tau_x\gamma), \quad \tau_x \text{ translation.}$$

$$= \int_{\mathbb{R}^n} d^n y\int_{\mathbb{R}^n}\gamma_L(d^n x)A(0,y,\tau_{xL}\gamma)\left(f(x+y/L)-f(x)\right)$$

- Independent jump : $A(x,y,\gamma) = a(x-y)$ Then

$$= \int_{\mathbb{R}^n} d^n y \int_{\mathbb{R}^n} \gamma_L(d^n x) a(y) \left(f(x+y/L) - f(x)\right)$$

- Expansion in large L

$$= \int_{\mathbb{R}^n} d^n y \int_{\mathbb{R}^n} \gamma_L(d^n x) A(0,y,\tau_{xL}\gamma) \Bigg(f(x) + \sum_{k=1}^{n} \partial_i f(x) y/L$$

$$+ \sum_{k,j=1}^{n} \partial_i \partial_j f(x) y_i y_j / L^2 + O(L^{-3}) - f(x) \Bigg)$$

$$= \int_{\mathbb{R}^n} \gamma_L(d^n x) \Bigg(L^{-1} \sum_{k=1}^{n} a_i^{(1)}(\tau_{xL}\gamma) \partial_i f(x)$$

$$+ L^{-2} \sum_{k,j=1}^{n} a_{i,j}^{(2)}(\tau_{xL}\gamma) \partial_i \partial_j f(x) + O(L^{-3}) \Bigg)$$

where $a_i^{(1)}(\gamma) = \int_{\mathbb{R}^n} d^n y A(0,y,\gamma) y_i$
and $a_{i,j}^{(2)}(\gamma) = \int_{\mathbb{R}^n} d^n y A(0,y,\gamma) y_i y_j$

- rewritten

$$= L^{-1}\sum_{k=1}^{n}\int_{\mathbb{R}^n}\gamma_L(d^nx)\partial_i f(x)a_i^{(1)}(\tau_{xL}\gamma)$$
$$+L^{-2}\sum_{k,j=1}^{n}\int_{\mathbb{R}^n}\gamma_L(d^nx)\partial_i\partial_j f(x)a_{i,j}^{(2)}(\tau_{xL}\gamma)+O(L^{-3})$$

- Independent jump: $A(x,y,\gamma)=a(x-y)$

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^n}\gamma_L(t,d^nx)f(x) &= L^{-1}\sum_{k=1}^{n}\int_{\mathbb{R}^n}\gamma_L(t,d^nx)\partial_i f(x)a_i^{(1)}\\ &\quad +L^{-2}\sum_{k,j=1}^{n}\int_{\mathbb{R}^n}\gamma_L(t,d^nx)\partial_i\partial_j f(x)a_{i,j}^{(2)}\\ &\quad +\frac{d}{dt}M(F,t)\end{aligned}$$

- In expectation

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^n}\rho_L(t,d^nx)f(x) &= \frac{d}{dt}\int_{\mathbb{R}^n}\langle\gamma_L(t,d^nx)\rangle f(x)\\ = \quad & L^{-1}\sum_{k=1}^{n}\int_{\mathbb{R}^n}\rho_L(t,d^nx)\partial_i f(x)a_i^{(1)}\\ & +L^{-2}\sum_{k,j=1}^{n}\int_{\mathbb{R}^n}\rho_L(t,d^nx)\partial_i\partial_j f(x)a_{i,j}^{(2)}+O(L^{-3})\end{aligned}$$

- On timescale L we get

$$\frac{d}{dt}\int_{\mathbb{R}^n}\rho(t,d^nx)f(x) \quad = \quad \sum_{k=1}^{n}\int_{\mathbb{R}^n}\rho(t,d^nx)\partial_i f(x)a_i^{(1)}$$

- If $a^{(1)}=0$ on timescale L^2 we get

$$\frac{d}{dt}\int_{\mathbb{R}^n}\rho(t,d^nx)f(x) \quad = \quad \sum_{k,j=1}^{n}\int_{\mathbb{R}^n}\rho_L(t,d^nx)\partial_i\partial_j f(x)a_{i,j}^{(2)}$$

- for each sample

$$\frac{d}{dt}\int_{\mathbb{R}^n}\gamma_L(t,d^nx)f(x) = L^{-1}\sum_{k=1}^{n}\int_{\mathbb{R}^n}\gamma_L(t,d^nx)\partial_i f(x)a_i^{(1)}$$

$$+L^{-2}\sum_{k,j=1}^{n}\int_{\mathbb{R}^n}\gamma_L(t,d^nx)\partial_i\partial_j f(x)a_{i,j}^{(2)} + O(L^{-3}) + \frac{d}{dt}M(F,t)$$

- If $a^{(1)} = 0$ on timescale L^2 we get for $L \to \infty$

$$P\left(\left|\int_{\mathbb{R}^n}\gamma_L(t,d^nx)f(x) - \int_{\mathbb{R}^n}\rho(t,d^nx)f(x)\right| \geq \varepsilon\right) = 0$$

Interacting

- In expectation

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}\rho_L(t,d^nx)f(x)=\frac{d}{dt}\int_{\mathbb{R}^n}\langle\gamma_L(t,d^nx)\rangle f(x)\\
=\quad&L^{-1}\sum_{k=1}^{n}\int_{\mathbb{R}^n}\partial_i f(x)\langle\gamma_L(t,d^nx)a_i^{(1)}(\tau_{xL}\gamma_t)\rangle\\
&+L^{-2}\sum_{k,j=1}^{n}\int_{\mathbb{R}^n}\partial_i\partial_j f(x)\langle\gamma_L(t,d^nx)a_{i,j}^{(2)}(\tau_{xL}\gamma_t)+O(L^{-3})
\end{aligned}
$$

- If $a^{(1)}=0$ on timescale L^2 we get

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n}\rho_L(t,d^nx)f(x)\\
=\quad&L^{-2}\sum_{k,j=1}^{n}\int_{\mathbb{R}^n}\partial_i\partial_j f(x)\langle\gamma_L(t,d^nx)a_{i,j}^{(2)}(\tau_{xL}\gamma_t)\rangle+O(L^{-3})
\end{aligned}
$$

- Type of closure problem

$$
\langle\gamma_L(t,d^nx)a_{i,j}^{(2)}(\tau_{xL}\gamma_t)\rangle=D(\rho_L)\qquad\text{What is } D(\rho_l)
$$

- Assume existence reversible spatial point process
- Example $A(0, y, \gamma) = A(0, -y, \gamma)$ (gradient system)
- Invariant spatial point processes: all Poisson π_z, $z \geq 0$
- Local equilibrium: large time, order L^2

$$\langle \int_{\mathbb{R}^n} g(x)\gamma_L(t, d^n x) a_{i,j}^{(2)}(\tau_{xL}\gamma(t, \cdot))\rangle \sim \int_{\mathbb{R}^n} g(x)\gamma_L(t, d^n x) \int a_{i,j}^{(2)}(\gamma)\pi$$

Define

$$D_{i,j}(z) = \int a_{i,j}^{(2)}(\gamma)\pi_z(d\gamma)$$

- Limiting equation

$$\frac{d}{dt}\int_{\mathbb{R}^n}\rho(t,d^nx)f(x) \quad = \quad \sum_{k,j=1}^{n}\int_{\mathbb{R}^n} D_{i,j}(\rho(t,x))\partial_i\partial_j f(x)\rho(t,x)d^n x$$

- If $A(0,y,\gamma) \neq A(0,-y,\gamma)$ then relation is given by variational principle (Green-Kubo)

- Then

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^n} f(x)\rho_L(t,dx) &= \frac{d}{dt}\langle \int_{\mathbb{R}^n} f(x)\gamma_L(dx)\rangle \qquad (3)\\ &= L^{-d}\frac{d}{dt}\int_{\mathbb{R}^n} f(x/L)\gamma(dx) \qquad (4)\end{aligned}$$

- Birth part

$$\begin{aligned} &L^{-d}\int_{\mathbb{R}^n} d^n y \int_{\mathbb{R}^n} \gamma(d^n x) B(x,y,\gamma) f(y/L)\\ &= \int_{\mathbb{R}^n} d^n y \int_{\mathbb{R}^n} \gamma_L(d^n x) B(xL,y,\gamma) f(y/L)\end{aligned}$$

- Translation invariance

$$B(x,y,\gamma) = B(0,y-x,\tau_x\gamma), \quad \tau_x \text{ translation.}$$

$$= \int_{\mathbb{R}^n} d^n y \int_{\mathbb{R}^n} \gamma_L(d^n x) B(0, xL+y, \tau_{xL}\gamma) f(x+y/L)$$

- expansion in L

$$= \int_{\mathbb{R}^n} \gamma_L(d^n x) \left(b^{(0)}(\tau_{xL}\gamma) f(x) + L^{-1} \sum_{k=1}^{n} b_i^{(1)}(\tau_{xL}\gamma) \partial_i f(x) \right.$$
$$\left. + L^{-2} \sum_{k,j=1}^{n} b_{i,j}^{(2)}(\tau_{xL}\gamma) \partial_i \partial_j f(x) + O(L^{-3}) \right)$$

where

$$\begin{aligned} b^{(0)}(\gamma) &= \int_{\mathbb{R}^n} d^n y B(0,y,\gamma) \\ b_i^{(1)}(\gamma) &= \int_{\mathbb{R}^n} d^n y B(0,y,\gamma) y_i \\ b_{i,j}^{(2)}(\gamma) &= \int_{\mathbb{R}^n} d^n y B(0,y,\gamma) y_i y_j \end{aligned}$$

Independent birth

- independent birth

$$B(x, y, \gamma) = b(x - y)$$

Then

$$= \int_{\mathbb{R}^n} d^n y \int_{\mathbb{R}^n} \gamma_L(d^n x) b(y) f(x + y/L)$$

- Expansion in large L

$$= \int_{\mathbb{R}^n} d^n y \int_{\mathbb{R}^n} \gamma_L(d^n x) b(y) \Bigg(f(x) + \sum_{k=1}^{n} \partial_i f(x) y/L$$

$$+ \sum_{k,j=1}^{n} \partial_i \partial_j f(x) y_i y_j / L^2 + O(L^{-3}) \Bigg)$$

$$= \int_{\mathbb{R}^n} \gamma_L(d^n x) \Bigg(b^{(0)} f(x) + L^{-1} \sum_{k=1}^{n} b_i^{(1)} \partial_i f(x)$$

$$+ L^{-2} \sum^{n} b_{i,j}^{(2)} \partial_i \partial_j f(x) + O(L^{-3}) \Bigg)$$

- $$= \int_{\mathbb{R}^n} \gamma_L(d^n x) \left(b^{(0)} f(x) + L^{-1} \sum_{k=1}^{n} b_i^{(1)} \partial_i f(x) \right. $$
$$\left. + L^{-2} \sum_{k,j=1}^{n} b_{i,j}^{(2)} \partial_i \partial_j f(x) + O(L^{-3}) \right)$$

  where

$$\begin{aligned} b^{(0)} &= \int_{\mathbb{R}^n} d^n y b(y) \\ b_i^{(1)} &= \int_{\mathbb{R}^n} d^n y b(y) y_i \\ b_{i,j}^{(2)} &= \int_{\mathbb{R}^n} d^n y b(y) y_i y_j \end{aligned}$$

Independent death

- idependent death

$$D(x,\gamma) = d^{(0)}$$

Then

$$= -\int_{\mathbb{R}^n} \gamma_L(d^n x) d^{(0)} f(x)$$

Altogether

$$\frac{d}{dt}\int_{\mathbb{R}^n} \gamma_L(d^n x) f(x) = (b^{(0)} - d^{(0)}) \int_{\mathbb{R}^n} \gamma_L(d^n x) f(x)$$

$$+L^{-1} \sum_{k=1}^{n} b_i^{(1)} \int_{\mathbb{R}^n} \gamma_L(d^n x) \partial_i f(x)$$

$$+L^{-2} \sum_{k,j=1}^{n} b_{i,j}^{(2)} \int_{\mathbb{R}^n} \gamma_L(d^n x) \partial_i \partial_j f(x) + O(L^{-3}) + \frac{d}{dt} M(F,t)$$

- In expectation

$$\frac{d}{dt}\int_{\mathbb{R}^n} d^n x \rho_L(t,x) f(x) = (b^{(0)} - d^{(0)}) \int_{\mathbb{R}^n} d^n x \rho_L(t,x) f(x)$$
$$+L^{-1} \sum_{k=1}^{n} b_i^{(1)} d^n x \rho_L(t,x) \partial_i f(x)$$
$$+L^{-2} \sum_{k,j=1}^{n} b_{i,j}^{(2)} \int_{\mathbb{R}^n} d^n x \rho_L(t,x) \partial_i \partial_j f(x) + O(L^{-3}) + 0$$

- No moment closure problem

Characteristics

- Martingale: $F(X_t) - F(X_0) - \int_0^t ds\, LF(X_s) =: M_t(F, X)$
- Generator

$$\begin{aligned} LF(\gamma) &= d \sum_{x \in \mathrm{supp}(\gamma)} \Big(F(\gamma - \delta_x) - F(\gamma) \Big) \\ &\quad + \int dy \sum_{x \in \mathrm{supp}(\gamma)} a(x-y) \Big(F(\gamma + \delta_y) - F(\gamma) \Big) \end{aligned}$$

- Compensator

$$\Big\langle M(F, X) \Big\rangle_t = \int_0^t ds\, \square_L(F,F)(X_s)$$

$$\begin{aligned} \square_L(F,F)(\gamma) &:= d \sum_{x \in \mathrm{supp}(\gamma)} \Big(F(\gamma - \delta_x) - F(\gamma) \Big)^2 \\ &\quad + \int dy \sum_{x \in \mathrm{supp}(\gamma)} a(x-y) \Big(F(\gamma + \delta_y) - F(\gamma) \Big)^2 \end{aligned}$$

Characteristics

- Martingale: $F(X_t) - F(X_0) - \int_0^t ds\, LF(X_s) =: M_t(F,X)$
- Generator

$$\begin{aligned} LF(\gamma) &= d \sum_{x \in \mathrm{supp}(\gamma)} \Big(F(\gamma - \delta_x) - F(\gamma) \Big) \\ &\quad + \int dy \sum_{x \in \mathrm{supp}(\gamma)} a(x-y) \Big(F(\gamma + \delta_y) - F(\gamma) \Big) \end{aligned}$$

- Compensator

$$\Big\langle M(F,X) \Big\rangle_t = \int_0^t ds\, \square_L(F,F)(X_s)$$

$$\begin{aligned} \square_L(F,F)(\gamma) &:= d \sum_{x \in \mathrm{supp}(\gamma)} \Big(F(\gamma - \delta_x) - F(\gamma) \Big)^2 \\ &\quad + \int dy \sum_{x \in \mathrm{supp}(\gamma)} a(x-y) \Big(F(\gamma + \delta_y) - F(\gamma) \Big)^2 \end{aligned}$$

Invariant measures

- density $\rho_t \to \begin{cases} +\infty & \text{for } d < \int dy\, a(y) \\ 0 & \text{for } d = \int dy\, a(y) \\ -\infty & \text{for } d > \int dy\, a(y) \end{cases}$
- Time development of correlation functions

$$\begin{aligned} \frac{\partial k_t}{\partial t}(\eta) &= |\eta| \left(\int dy a(y) - d \right) k_t(\eta) \\ &+ \sum_{x \in \eta} \int dy a(x-y) \left[k_t(\eta \cup y \setminus x) - k_t(\eta) \right] \\ &+ \sum_{x_1 \in \eta} \sum_{x_2 \in \eta} a(x_1 - x_2) k_t(\eta \setminus x_1) \end{aligned}$$

- Critical value: $d = \int dy\, a(y)$
- Characterization of invariant measure via correlation (cummulants)
- Bounds $O((n!)^2)$. Clustering

Invariant measures

- density $\rho_t \to \begin{cases} +\infty & \text{for } d < \int dy\, a(y) \\ 0 & \text{for } d = \int dy\, a(y) \\ -\infty & \text{for } d > \int dy\, a(y) \end{cases}$
- Time development of correlation functions

$$\begin{aligned} \frac{\partial k_t}{\partial t}(\eta) &= |\eta| \left(\int dy a(y) - d \right) k_t(\eta) \\ &+ \sum_{x \in \eta} \int dy a(x-y) \left[k_t(\eta \cup y \setminus x) - k_t(\eta) \right] \\ &+ \sum_{x_1 \in \eta} \sum_{x_2 \in \eta} a(x_1 - x_2) k_t(\eta \setminus x_1) \end{aligned}$$

- Critical value: $d = \int dy\, a(y)$
- Characterization of invariant measure via correlation (cummulants)
- Bounds $O((n!)^2)$. Clustering

Invariant measures

- density $\rho_t \to \begin{cases} +\infty & \text{for } d < \int dy\, a(y) \\ 0 & \text{for } d = \int dy\, a(y) \\ -\infty & \text{for } d > \int dy\, a(y) \end{cases}$
- Time development of correlation functions

$$\begin{aligned} \frac{\partial k_t}{\partial t}(\eta) \quad &= \\ &+ \sum_{x \in \eta} \int dy a(x-y) \left[k_t(\eta \cup y \setminus x) - k_t(\eta)\right] \\ &+ \sum_{x_1 \in \eta} \sum_{x_2 \in \eta} a(x_1 - x_2) k_t(\eta \setminus x_1) \end{aligned}$$

- Critical value: $d = \int dy\, a(y)$
- Characterization of invariant measure via correlation (cummulants)
- Bounds $O((n!)^2)$. Clustering

Invariant measures

- LLN:

$$\varepsilon^n \sum_{x\in\gamma} \varphi(\varepsilon x) \to \rho \int dy \varphi(y)$$

- CLT:

$$\varepsilon^{n/2+1} \left(\sum_{x\in\gamma} \varphi(\varepsilon x) - \varepsilon^{-n} \rho \int dy \varphi(y) \right)$$

converges to Gauss measure on $\mathcal{S}'(\mathbb{R}^n)$ with covariance

$$2\rho \left(\sum_{i,j=1}^{n} \int_{\mathbb{R}^n} y_i y_j a(y) dy \partial_i \partial_j \right)^{-1}$$

Characteristics

- $X^{(\varepsilon)}$ fulfills

$$\langle \varphi, X_t^{(\varepsilon)}\rangle - \langle \varphi, X_0^{(\varepsilon)}\rangle - \int_0^t ds L^{(\varepsilon)}\langle \varphi, \cdot\rangle (X_s^{(\varepsilon)}) =: M_t^{(\varepsilon)}(\langle \varphi, \cdot\rangle, X^{(\varepsilon)})$$

is a martingale, where

$$L^{(\varepsilon)}\langle \varphi, \cdot\rangle = \langle A_\varepsilon \varphi, \gamma\rangle$$

with

$$A_\varepsilon \varphi(x) := -d\varepsilon^{-2}\varphi(x) + \varepsilon^{-2}\int dy \varepsilon^{-n} a(\varepsilon^{-1}(y-x))\varphi(y)$$

- Compensator

$$\langle M^{(\varepsilon)}(\langle \varphi, \cdot\rangle, X^{(\varepsilon)})\rangle_t = \int_0^t ds \Big\langle d\,\varphi^2 + \int dy \varepsilon^{-n} a(\varepsilon^{-1}(y-\cdot))\varphi^2(y), X_s^{(\varepsilon)}\Big\rangle$$

Characteristics

- $X^{(\varepsilon)}$ fulfills

  $$\langle \varphi, X_t^{(\varepsilon)} \rangle - \langle \varphi, X_0^{(\varepsilon)} \rangle - \int_0^t ds L^{(\varepsilon)} \langle \varphi, \cdot \rangle (X_s^{(\varepsilon)}) =: M_t^{(\varepsilon)}(\langle \varphi, \cdot \rangle, X^{(\varepsilon)})$$

  is a martingale, where

  $$L^{(\varepsilon)} \langle \varphi, \cdot \rangle = \langle A_\varepsilon \varphi, \gamma \rangle$$

  with

  $$A_\varepsilon \varphi(x) := -d\varepsilon^{-2} \varphi(x) + \varepsilon^{-2} \int dy \varepsilon^{-n} a(\varepsilon^{-1}(y-x)) \varphi(y)$$

- Compensator

$$\left\langle M^{(\varepsilon)}(\langle \varphi, \cdot \rangle, X^{(\varepsilon)}) \right\rangle_t = \int_0^t ds \left\langle d\, \varphi^2 + \int dy \varepsilon^{-n} a(\varepsilon^{-1}(y - \cdot)) \varphi^2(y), X_s^{(\varepsilon)} \right\rangle$$

Characteristics

- $X^{(\varepsilon)}$ fulfills

  $$\langle \varphi, X_t^{(\varepsilon)} \rangle - \langle \varphi, X_0^{(\varepsilon)} \rangle - \int_0^t ds L^{(\varepsilon)} \langle \varphi, \cdot \rangle (X_s^{(\varepsilon)}) =: M_t^{(\varepsilon)}(\langle \varphi, \cdot \rangle, X^{(\varepsilon)})$$

  is a martingale, where

  $$L^{(\varepsilon)} \langle \varphi, \cdot \rangle = \langle A_\varepsilon \varphi, \gamma \rangle$$

  with

  $$A_\varepsilon \varphi(x) := -d\varepsilon^{-2} \varphi(x) + \varepsilon^{-2} \int dy \varepsilon^{-n} a(\varepsilon^{-1}(y - x)) \varphi(y)$$

- Compensator

$$\langle M^{(\varepsilon)}(\langle \varphi, \cdot \rangle, X^{(\varepsilon)}) \rangle_t = \int_0^t ds \Big\langle d\, \varphi^2 + \int dy \varepsilon^{-n} a(\varepsilon^{-1}(y - \cdot)) \varphi^2(y), X_s^{(\varepsilon)} \Big\rangle$$

Characteristics

- $X^{(\varepsilon)}$ fulfills

$$\langle \varphi, X_t^{(\varepsilon)} \rangle - \langle \varphi, X_0^{(\varepsilon)} \rangle - \int_0^t ds L^{(\varepsilon)} \langle \varphi, \cdot \rangle (X_s^{(\varepsilon)}) =: M_t^{(\varepsilon)}(\langle \varphi, \cdot \rangle, X^{(\varepsilon)})$$

is a martingale, where

$$L^{(\varepsilon)} \langle \varphi, \cdot \rangle = \langle A_\varepsilon \varphi, \gamma \rangle$$

with

$$A_\varepsilon \varphi(x) := -d\varepsilon^{-2} \varphi(x) + \varepsilon^{-2} \int dy \varepsilon^{-n} a(\varepsilon^{-1}(y-x)) \varphi(y)$$

- Compensator

$$\left\langle M^{(\varepsilon)}(\langle \varphi, \cdot \rangle, X^{(\varepsilon)}) \right\rangle_t = \int_0^t ds \left\langle d\, \varphi^2 + \int dy \varepsilon^{-n} a(\varepsilon^{-1}(y-\cdot)) \varphi^2(y), X_s^{(\varepsilon)} \right\rangle$$

Heuristic

- Generator

$$A_\varepsilon\varphi(x) := -d\varepsilon^{-2}\varphi(x) + \varepsilon^{-2}\int dya(y)\varphi(x+\varepsilon y)$$

- Critical scaling: $d = \int dya(y)$
- Generator

$$\begin{aligned} A_\varepsilon\varphi(x) &= \varepsilon^{-2}\int dya(y)\big(\varphi(x+\varepsilon y)-\varphi(x)\big) \\ &= \sum_{i,j=1}^{d}\Big(\int dya(y)y_iy_j\Big)\partial_i\partial_j\varphi(x) + o(\varepsilon) \end{aligned}$$

- Compensator

$$\big\langle M^{(\varepsilon)}(\langle\varphi,\cdot\rangle, X^{(\varepsilon)})\big\rangle_t = \frac{1}{2}\int_0^t ds\int dya(y)\langle\varphi^2+\varphi^2(\cdot+\varepsilon y), X_s^{(\varepsilon)}\rangle$$

Heuristic

- Generator

$$A_\varepsilon \varphi(x) := -d\varepsilon^{-2}\varphi(x) + \varepsilon^{-2}\int dy a(y)\varphi(x+\varepsilon y)$$

- Critical scaling: $d = \int dy a(y)$
- Generator

$$\begin{aligned} A_\varepsilon \varphi(x) &= \varepsilon^{-2}\int dy a(y)\big(\varphi(x+\varepsilon y) - \varphi(x)\big) \\ &= \sum_{i,j=1}^{d} \Big(\int dy a(y) y_i y_j\Big)\, \partial_i \partial_j \varphi(x) + o(\varepsilon) \end{aligned}$$

- Compensator

$$\big\langle M^{(\varepsilon)}(\langle \varphi, \cdot\rangle, X^{(\varepsilon)})\big\rangle_t = \frac{1}{2}\int_0^t ds \int dy a(y) \langle \varphi^2 + \varphi^2(\cdot + \varepsilon y), X_s^{(\varepsilon)}\rangle$$

Heuristic

- Generator

$$A_\varepsilon\varphi(x) := -d\varepsilon^{-2}\varphi(x) + \varepsilon^{-2}\int dy a(y)\varphi(x+\varepsilon y)$$

- Critical scaling: $d = \int dy a(y)$
- Generator

$$\begin{aligned} A_\varepsilon\varphi(x) &= \varepsilon^{-2}\int dy a(y)\big(\varphi(x+\varepsilon y)-\varphi(x)\big) \\ &= \sum_{i,j=1}^{d} \Big(\int dy a(y) y_i y_j\Big)\, \partial_i\partial_j\varphi(x) + o(\varepsilon) \end{aligned}$$

- Compensator

$$\left\langle M^{(\varepsilon)}(\langle\varphi,\cdot\rangle, X^{(\varepsilon)})\right\rangle_t = \frac{1}{2}\int_0^t ds\int dy a(y)\langle\varphi^2+\varphi^2(\cdot+\varepsilon y), X_s^{(\varepsilon)}\rangle$$

Heuristic

- Generator

$$A_\varepsilon \varphi(x) := -d\varepsilon^{-2}\varphi(x) + \varepsilon^{-2}\int dy a(y)\varphi(x+\varepsilon y)$$

- Critical scaling: $d = \int dy a(y)$
- Generator

$$\begin{aligned} A_\varepsilon \varphi(x) &= \varepsilon^{-2}\int dy a(y)\big(\varphi(x+\varepsilon y) - \varphi(x)\big) \\ &= \sum_{i,j=1}^{d} \Big(\int dy a(y) y_i y_j\Big)\, \partial_i\partial_j \varphi(x) + o(\varepsilon) \end{aligned}$$

- Compensator

$$\langle M^{(\varepsilon)}(\langle \varphi, \cdot\rangle, X^{(\varepsilon)})\rangle_t = \frac{1}{2}\int_0^t ds \int dy a(y) \langle \varphi^2 + \varphi^2(\cdot + \varepsilon y), X_s^{(\varepsilon)}\rangle$$

Heuristic

- Generator

$$A_\varepsilon \varphi(x) := -d\varepsilon^{-2}\varphi(x) + \varepsilon^{-2}\int dy a(y)\varphi(x+\varepsilon y)$$

- Critical scaling: $d = \int dy a(y)$
- Generator

$$\begin{aligned} A_\varepsilon \varphi(x) &= \varepsilon^{-2}\int dy a(y)\big(\varphi(x+\varepsilon y) - \varphi(x)\big) \\ &= \sum_{i,j=1}^{d} \Big(\int dy a(y) y_i y_j \Big)\, \partial_i \partial_j \varphi(x) + o(\varepsilon) \end{aligned}$$

- Compensator

$$\big\langle M^{(\varepsilon)}(\langle \varphi, \cdot \rangle, X^{(\varepsilon)}) \big\rangle_t = \frac{1}{2}\int_0^t ds \int dy a(y) \langle \varphi^2 + \varphi^2(\cdot + \varepsilon y), X_s^{(\varepsilon)} \rangle$$

Heuristic

- Generator

$$A_\varepsilon \varphi(x) := -d\varepsilon^{-2}\varphi(x) + \varepsilon^{-2}\int dy a(y)\varphi(x+\varepsilon y)$$

- Critical scaling: $d = \int dy a(y)$
- Generator

$$\begin{aligned} A_\varepsilon \varphi(x) &= \varepsilon^{-2}\int dy a(y)\big(\varphi(x+\varepsilon y) - \varphi(x)\big) \\ &= \sum_{i,j=1}^{d} \Big(\int dy a(y) y_i y_j \Big) \partial_i \partial_j \varphi(x) + o(\varepsilon) \end{aligned}$$

- Compensator

$$\begin{aligned} \big\langle M^{(\varepsilon)}(\langle \varphi, \cdot \rangle, X^{(\varepsilon)}) \big\rangle_t &= \frac{1}{2}\int_0^t ds \int dy a(y) \langle \varphi^2 + \varphi^2(\cdot + \varepsilon y), X_s^{(\varepsilon)} \rangle \\ &= \int_0^t ds\, 2 \int dy a(y) \langle \varphi^2, X_s^{(\varepsilon)} \rangle + o(\varepsilon) \end{aligned}$$

Main result

Theorem

Let $X_0^{(\varepsilon)} \in \Gamma^{(\varepsilon)}(X)$ *with* $X_0^{(\varepsilon)} \to \nu$.

Then $(X^{(\varepsilon)})_\varepsilon$ *converges weakly w.r.t. Skorokhod topology to a unique measure valued càdlàg process* X *such that*

$$M(X)_t := \langle \varphi, X_t \rangle - \langle \varphi, X_0 \rangle - \int_0^t ds \langle A\varphi, X_s \rangle$$

is a martingale, where $A\varphi := \frac{1}{2} \sum_{i,j=1}^d \left(\int dy a(y) y_i y_j \right) \partial_i \partial_j$ *and*

$$\langle M(X) \rangle_t = \int_0^t ds \, 2 \int dy a(y) \langle \varphi^2, X_s \rangle$$

Main result

Theorem

Let $X_0^{(\varepsilon)} \in \Gamma^{(\varepsilon)}(X)$ *with* $X_0^{(\varepsilon)} \to \nu$.
Then $(X^{(\varepsilon)})_\varepsilon$ *converges weakly w.r.t. Skorokhod topology to a unique measure valued càdlàg process* X
such that

$$M(X)_t := \langle \varphi, X_t \rangle - \langle \varphi, X_0 \rangle - \int_0^t ds \langle A\varphi, X_s \rangle$$

is a martingale, where $A\varphi := \frac{1}{2} \sum_{i,j=1}^d \left(\int dy a(y) y_i y_j \right) \partial_i \partial_j$
and

$$\langle M(X) \rangle_t = \int_0^t ds\, 2 \int dy a(y) \langle \varphi^2, X_s \rangle$$

Main result

Theorem

Let $X_0^{(\varepsilon)} \in \Gamma^{(\varepsilon)}(X)$ *with* $X_0^{(\varepsilon)} \to \nu$.
Then $(X^{(\varepsilon)})_\varepsilon$ *converges weakly w.r.t. Skorokhod topology to a unique measure valued càdlàg process* X
such that

$$M(X)_t := \langle \varphi, X_t \rangle - \langle \varphi, X_0 \rangle - \int_0^t ds \langle A\varphi, X_s \rangle$$

is a martingale, where $A\varphi := \frac{1}{2} \sum_{i,j=1}^d \left(\int dy a(y) y_i y_j \right) \partial_i \partial_j$
and

$$\langle M(X) \rangle_t = \int_0^t ds \, 2 \int dy a(y) \langle \varphi^2, X_s \rangle$$

Main result

Theorem

Let $X_0^{(\varepsilon)} \in \Gamma^{(\varepsilon)}(X)$ *with* $X_0^{(\varepsilon)} \to \nu$.
Then $(X^{(\varepsilon)})_\varepsilon$ *converges weakly w.r.t. Skorokhod topology to a unique measure valued càdlàg process* X
such that

$$M(X)_t := \langle \varphi, X_t \rangle - \langle \varphi, X_0 \rangle - \int_0^t ds \langle A\varphi, X_s \rangle$$

is a martingale, where $A\varphi := \frac{1}{2} \sum_{i,j=1}^d \left(\int dy a(y) y_i y_j \right) \partial_i \partial_j$
and

$$\left\langle M(X) \right\rangle_t = \int_0^t ds\, 2 \int dy a(y) \langle \varphi^2, X_s \rangle$$

Lattice vs. continuum

- Lattice version: R. Durrett and E. A. Perkins, ’99
  - Same limiting process
  - Finite measure as initial condition
  - Artefacts from discrete structure
  - Slightly non-critical
- Continuous version
  - Infinite measures as initial condition
  - new phenomenology: additional invariant initial condition
  - Quasi solvable
  - Exact critical value for fixed ε
  - Explicit formula for invariant measure
  - Instable

Lattice vs. continuum

- Lattice version: R. Durrett and E. A. Perkins, ’99
    - Same limiting process
    - Finite measure as initial condition
    - Artefacts from discrete structure
    - Slightly non-critical
- Continuous version
    - Infinite measures as initial condition
    - new phenomenology: additional invariant initial condition
    - Quasi solvable
    - Exact critical value for fixed ε
    - Explicit formula for invariant measure
    - Instable

Scheme of proof

- Bound uniform in $0 \leq t \leq T$ and ε of

$$\langle \varphi, X_t^{(\varepsilon)} \rangle$$

- Explicit scaling limit for $\langle \varphi, X_t^{(\varepsilon)} \rangle$.
- Control of process up to times $\varepsilon^{-2}T$.
- Compactness in weak topology w.r.t. Skorokhod topology
- Martingale representation, explicit formula for quadratic variation
- Convergence of generators

$$\mathbb{E}_{X_0^{(\varepsilon)}} \int_0^T ds \left| \langle A_\varepsilon \varphi, X_s^{(\varepsilon)} \rangle - \langle A \varphi, X_s^{(\varepsilon)} \rangle \right| \to 0$$

- Characterization by quadratic variation

Scheme of proof

- Bound uniform in $0 \leq t \leq T$ and ε of

$$\langle \varphi, X_t^{(\varepsilon)} \rangle$$

- Explicit scaling limit for $\langle \varphi, X_t^{(\varepsilon)} \rangle$.
- Control of process up to times $\varepsilon^{-2} T$.
- Compactness in weak topology w.r.t. Skorokhod topology
- Martingale representation, explicit formula for quadratic variation
- Convergence of generators

$$\mathbb{E}_{X_0^{(\varepsilon)}} \int_0^T ds \left| \langle A_\varepsilon \varphi, X_s^{(\varepsilon)} \rangle - \langle A \varphi, X_s^{(\varepsilon)} \rangle \right| \to 0$$

- Characterization by quadratic variation

Scheme of proof

- Bound uniform in $0 \leq t \leq T$ and ε of

$$\langle \varphi, X_t^{(\varepsilon)} \rangle$$

- Explicit scaling limit for $\langle \varphi, X_t^{(\varepsilon)} \rangle$.
- Control of process up to times $\varepsilon^{-2}T$.
- Compactness in weak topology w.r.t. Skorokhod topology
- Martingale representation, explicit formula for quadratic variation
- Convergence of generators

$$\mathbb{E}_{X_0^{(\varepsilon)}} \int_0^T ds \left| \langle A_\varepsilon \varphi, X_s^{(\varepsilon)} \rangle - \langle A \varphi, X_s^{(\varepsilon)} \rangle \right| \to 0$$

- Characterization by quadratic variation

Scheme of proof

- Bound uniform in $0 \leq t \leq T$ and ε of

$$\langle \varphi, X_t^{(\varepsilon)} \rangle$$

- Explicit scaling limit for $\langle \varphi, X_t^{(\varepsilon)} \rangle$.
- Control of process up to times $\varepsilon^{-2}T$.
- Compactness in weak topology w.r.t. Skorokhod topology
- Martingale representation, explicit formula for quadratic variation
- Convergence of generators

$$\mathbb{E}_{X_0^{(\varepsilon)}} \int_0^T ds \left| \langle A_\varepsilon \varphi, X_s^{(\varepsilon)} \rangle - \langle A \varphi, X_s^{(\varepsilon)} \rangle \right| \to 0$$

- Characterization by quadratic variation

Scheme of proof

- Bound uniform in $0 \leq t \leq T$ and ε of

$$\langle \varphi, X_t^{(\varepsilon)} \rangle$$

- Explicit scaling limit for $\langle \varphi, X_t^{(\varepsilon)} \rangle$.
- Control of process up to times $\varepsilon^{-2}T$.
- Compactness in weak topology w.r.t. Skorokhod topology
- Martingale representation, explicit formula for quadratic variation
- Convergence of generators

$$\mathbb{E}_{X_0^{(\varepsilon)}} \int_0^T ds \left| \langle A_\varepsilon \varphi, X_s^{(\varepsilon)} \rangle - \langle A\varphi, X_s^{(\varepsilon)} \rangle \right| \to 0$$

- Characterization by quadratic variation

Scheme of proof

- Bound uniform in $0 \leq t \leq T$ and ε of

$$\langle \varphi, X_t^{(\varepsilon)} \rangle$$

- Explicit scaling limit for $\langle \varphi, X_t^{(\varepsilon)} \rangle$.
- Control of process up to times $\varepsilon^{-2} T$.
- Compactness in weak topology w.r.t. Skorokhod topology
- Martingale representation, explicit formula for quadratic variation
- Convergence of generators

$$\mathbb{E}_{X_0^{(\varepsilon)}} \int_0^T ds \left| \langle A_\varepsilon \varphi, X_s^{(\varepsilon)} \rangle - \langle A \varphi, X_s^{(\varepsilon)} \rangle \right| \to 0$$

- Characterization by quadratic variation

Scheme of proof

- Bound uniform in $0 \leq t \leq T$ and ε of

$$\langle \varphi, X_t^{(\varepsilon)} \rangle$$

- Explicit scaling limit for $\langle \varphi, X_t^{(\varepsilon)} \rangle$.
- Control of process up to times $\varepsilon^{-2} T$.
- Compactness in weak topology w.r.t. Skorokhod topology
- Martingale representation, explicit formula for quadratic variation
- Convergence of generators

$$\mathbb{E}_{X_0^{(\varepsilon)}} \int_0^T ds \left| \langle A_\varepsilon \varphi, X_s^{(\varepsilon)} \rangle - \langle A \varphi, X_s^{(\varepsilon)} \rangle \right| \to 0$$

- Characterization by quadratic variation

Further development

- Hydrodynamic limit:

$$\varepsilon^n \int_{\mathbb{R}^n} \varphi(\varepsilon x) X_{\varepsilon^{-2}T}(dx)$$

- Critical fluctuations at non-zero density