Random Evolutions in Continuum

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AGENDA

1. Introduction: individual based models in continuum

2. Mathematical framework

3. Markov evolutions in configuration spaces

4. Stochastic individual based models (IBM)

4.1 IBM in spatial ecology

4.2 IBM in socio-economic systems

4.3 Multi-spices models

5. Spatially heterogeneous models

6. Mutation-selection models

7. Other topics in IBM

MACRO and MICRO models

PDE and SPDE

as macroscopic dynamical descriptions coming from
microscopic models via

- scaling limits for dynamics (hydrodynamic, Vlasov, Landau etc.)
- scaling of fluctuations (equilibrium or non-equilibrium)
- closure of (infinite linear) moment systems
- hierarchical chains (BBGKY etc.)
- heuristic arguments

....

MACRO and MICRO models

Some qualitative predictions
based on the use of PDE and SPDE

may be considered as approximations
to more detailed behaviors of microscopic systems,

which are mathematical models of the real world notions.

All models are caricatures of reality.

Mark Kac

$\mathrm{MICRO} \Rightarrow \mathrm{MACRO}$?

$\mathrm{MACRO} \Rightarrow \mathrm{MICRO}$?

Inverse problems, calibration, data assimilation, etc.

Complex Systems: BIO sciences

Simon A. Levin (Princeton)

"Complex Adaptive Systems: Exploring the Known, the Unknown and the Unknowable", Bull. AMS, 2002:

(1) diversity and individuality of components

(2) localized interactions among components

(3) the outcomes of interactions used for replication or enhancement of components

Complex Systems: SOCIO sciences

Philip Ball "Critical Mass" (2004): physics of society

– Complexity theory seeks to understand how the order and stability arise from the interactions of many agents

– We can make predictions about society even in the face of individual free will, and perhaps even illuminate the limits of that free will

– It is a science of humans collective behavior

Thomas Hobbes, "Leviaphan" (1651):
We must ask not just how things happen in society,
but why.

Statistical Mechanics for Complex Systems

R.GOMORY:

The central problem is to develop an appropriate statistical mechanics that allows one to separate the knowable unknown from the truly unknowable.

Such mechanics will have to deal with heterogeneous ensembles of interacting agents and with the continual refreshment of that ensemble by novel and unpredictable types.

Statistical Mechanics for Complex Systems

The shift from Newtonian determinism to statistical science is what makes a physics of society possible.

SOCIETY ITSELF IS FUNDAMENTALLY A STATISTICAL PHENOMENON.

Statistical Mechanics for Physics

Equilibrium StatPhys

Non-equilibrium StatPhys

Hamiltonian dynamics

Stochastic dynamics
(Glauber, Kawasaki, Metropolis, ...)

$$\boxed{\text{(Math. StatPhys)} \subset \text{(ID Analysis)}}$$

Interacting Particle Systems

IPS as models in

physics (gases, fluids, condensed matter)

chemical kinetics

population biology, ecology (individual based models=IBM)

sociology, economics (agent based models=ABM)

LATTICE (OR) (AND) (VS.) CONTINUOUS

Individual Based Models

RICHARD LAW ET AL., *Ecology*, **84**(2003):
IBM is a stochastic (Markov) process with events comprising

birth,

death,

and movement.

Ecological models:

Bolker/Pacala, 1997, ...
Dieckmann/Law/Metz, 2000, ...
....
Meleard et al., 2004
Birch/Young, 2006
Kondratiev/Srorokhod, 2006
Finkelshtein/Kondratiev/Kutovyi, 2007-2009

We will consider

INDIVIDUAL BASED DYNAMICAL COMPETITION MODELS

RATHER THAN

COEXISTENCE REGULATION MECHANISMS,
C.F.,
K/Minlos/Zhizhina, '07 (economics)
K/Kuna/Ohlerich, '07 (genetics)

Mathematics and real world models

”Richtiges Auffassen einer Sache und
Missverstehen der gleichen Sache
schlissen einander nich vollständig aus”.

F.Kafka

(Right understanding of a notion and its wrong understanding do not exclude each other completely)

Mathematics and real world models

From a speech of I.M.Gelfand at the Royal East Research, September 3, 2003:

"Mathematics is a language.
Mathematics is an adequate language in many areas such as physics, engineering, biology. This is a very important notion an adequate language...
The language of mathematics allows us to organize a lot of things".

General facts and notations

The configuration space:

$$\Gamma := \left\{ \gamma \subset \mathbb{R}^d \middle| \, |\gamma \cap \Lambda| < \infty \text{ for all compact } \Lambda \subset \mathbb{R}^d \right\}.$$

($| \cdot |$ cardinality of set).

Vague topology on Γ:
the weakest topology s.t. all functions

$$\Gamma \ni \gamma \mapsto \sum_{x \in \gamma} f(x) = \langle f, \gamma \rangle \in \mathbb{R}$$

are continuous for all $f \in C_0(\mathbb{R}^d)$.

Γ is a Polish space.

n-point configuration space:

$$\Gamma^{(n)} := \left\{\eta \subset \mathbb{R}^d \mid |\eta| = n\right\}, \quad n \in \mathbb{N}_0.$$

The space of finite configurations:

$$\Gamma_0 := \bigsqcup_{n\in\mathbb{N}_0} \Gamma^{(n)}.$$

K-transform

Combinatorial Fourier transform (Lenard; Kondratiev/ Kuna): additive type observables

$$KG(\gamma) := \sum_{\xi \Subset \gamma} G(\xi), \quad \gamma \in \Gamma$$

for

$$G : \Gamma_0 \to \mathbb{R}$$

(quasi-observables)

$$K^{-1}F(\eta) := \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} F(\xi), \quad \eta \in \Gamma_0$$

Convolution (Kondratiev/Kuna):

$$(G_1 \star G_2)(\eta) := \sum_{(\xi_1,\xi_2,\xi_3)\in\mathcal{P}^3_\emptyset(\eta)} G_1(\xi_1 \cup \xi_2) G_2(\xi_2 \cup \xi_3),$$

with property

$$K(G_1 \star G_2) = KG_1 \cdot KG_2$$

Correlation measure

$\mathcal{M}^1_{\mathrm{fm}}(\Gamma) \quad = \quad$ *probability measures* with finite local moments.

$\mathcal{M}_{\mathrm{lf}}(\Gamma_0) \quad = \quad$ *locally finite measures* on Γ_0.

One can define
$K^* : \mathcal{M}^1_{\mathrm{fm}}(\Gamma) \to \mathcal{M}_{\mathrm{lf}}(\Gamma_0)$:

$\forall \mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$, $G \in B_{\mathrm{bs}}(\Gamma_0)$

$$\int_\Gamma KG(\gamma)\mu(d\gamma) = \int_{\Gamma_0} G(\eta)\,(K^*\mu)(d\eta).$$

$$\rho_\mu := K^*\mu$$

is called the *correlation measure*.

Theorem

Let $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ *be given. For any* $G \in L^1(\Gamma_0, \rho_\mu)$ *we define*

$$KG(\gamma) := \sum_{\eta \Subset \gamma} G(\eta),$$

where the later series is μ*-a.s. absolutely convergent. Furthermore, we have* $KG \in L^1(\Gamma, \mu)$,

$$\int_{\Gamma_0} G(\eta)\, \rho_\mu(d\eta) = \int_\Gamma (KG)(\gamma)\, \mu(d\gamma).$$

Lebesgue-Poisson measure

$\sigma \quad = \quad$ Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

For any $n \in \mathbb{N}$ measure $\sigma^{\otimes n}$ can be considered on $\widetilde{(\mathbb{R}^d)^n}$.

$\sigma^{(n)} \quad = \quad$ projection on $\Gamma^{(n)}$.

The **Lebesgue-Poisson measure** λ_z, $z > 0$ on Γ_0:

$$\lambda_z := \sum_{n=0}^{\infty} \frac{z^n}{n!} \sigma^{(n)}.$$

Poisson measure

The Poisson measure π_z on $(\Gamma, \mathcal{B}(\Gamma))$ is given as the projective limit of the family of measures $\{\pi_z^\Lambda\}_{\Lambda \in \mathcal{B}_b(\mathbb{R}^d)}$, where π_z^Λ is the measure on Γ_Λ defined by

$$\pi_z^\Lambda := e^{-z\sigma(\Lambda)}\lambda_z.$$

$$\int_\Gamma e^{\langle f,\gamma\rangle} d\pi_z(\gamma) = \exp\{\int_{\mathbb{R}^d} (e^{f(x)} - 1) z dx\}.$$

$$\int_\Gamma \sum_{x\in\gamma} F(x,\gamma) d\pi_z(\gamma) = \int_\Gamma \int_{\mathbb{R}^d} F(x,\gamma) z dx d\pi_z(\gamma)$$

Correlation functions

A measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ is called

locally absolutely continuous

w.r.t. π_z iff $\mu_\Lambda := \mu \circ p_\Lambda^{-1}$

is absolutely continuous with respect to $\pi_z^\Lambda = \pi_z \circ p_\Lambda^{-1}$
for all $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$.

In this case $\rho_\mu := K^*\mu$ is absolutely continuous w.r.t λ_z.

$$k_\mu(\eta) := \frac{d\rho_\mu}{d\lambda_z}(\eta),\ \eta \in \Gamma_0.$$

$$k_\mu^{(n)} : (\mathbb{R}^d)^n \longrightarrow \mathbb{R}^+$$

$$k_\mu^{(n)}(x_1, \ldots, x_n) :=$$

$$k_\mu(\{x_1, \ldots, x_n\})$$

correlation functions.

Definition

A measure $\rho \in \mathcal{M}_{lf}(\Gamma_0)$ is called positive definite if

$$\int_{\Gamma_0} (G \star \overline{G})(\eta)\rho(d\eta) \geq 0, \;\; \forall G \in B_{bs}(\Gamma_0),$$

where $\overline{G}$ is a complex conjugate of G. The measure ρ is called normalized iff $\rho(\{\emptyset\}) = 1$.

Theorem (Kondratiev/Kuna)

Let $\rho \in \mathcal{M}_{lf}(\Gamma_0)$ *be given. Assume that* ρ *is positive definite, normalized and that for each bounded open* $\Lambda \subset \mathbb{R}^d$, *for every* $C > 0$ *there exists* $D_{\Lambda, C} > 0$ *s.t.*

$$\rho(\Gamma_\Lambda^n) \leq D_{\Lambda, C} C^n, \quad n \in \mathbb{N}_0.$$

Then there exists a unique measure $\mu \in \mathcal{M}^1_{fm}(\Gamma)$ *with* $\rho = K^\star \mu$.

Remark: A sufficient condition for the bound in the theorem:

$$k^{(n)} \leq C^n (n!)^{1-\varepsilon}.$$

B-A-D generators

Birth-and-death processes in continuum with generators

$$(LF)(\gamma) = \sum_{x\in\gamma} d(x,\gamma\setminus x)[F(\gamma\setminus x) - F(\gamma)] +$$
$$\int_{\mathbb{R}^d} b(x,\gamma)[F(\gamma\cup x) - F(\gamma)]\, dx$$

Some ecological processes are specified by:

$b(x,\gamma) = \sum_{y\in\gamma} B_y(x,\gamma)$

$b(x,\emptyset) = 0$

Existence problem

$$\left.\begin{array}{l}\text{K/Lytvynov}\\ \text{K/Lytvynov/Röckner}\end{array}\right\}\text{Dirichlet forms}$$

$$\left.\begin{array}{l}\text{K/Kutoviy/Zhizhina}\\ \text{K/Kutoviy/Minlos}\end{array}\right\}\text{Glauber dynamics}$$

$$\left.\begin{array}{l}\text{Holley/Stroock}\\ \text{Ferrari/ Garcia}\\ \text{K/Skorokhod}\\ \text{Garcia/Kurtz}\\ \text{Finkelshtein/K/Kutoviy}\end{array}\right\}\text{particular classes}$$

Correlation equations

moment equations $=$ hierarchical equations

$\mu_0 =$ initial distribution

$X_t^{\mu_0} \in \Gamma$ Markov process with initial distribution μ_0

$\mu_t \in \mathcal{M}^1(\Gamma)$ distribution at time $t > 0$

Correlation functions

$f^{(n)}(x_1, \ldots, x_n)$ symmetric function on $\mathbb{R}^{nd}$,
$\gamma = \{x_1, x_2, \ldots\} \subset \mathbb{R}^d$

$$\int_\Gamma \sum_{\{x_{i_1}, \ldots, x_{i_n}\} \subset \gamma} f^{(n)}(x_{i_1}, \ldots, x_{i_n})\, d\mu(\gamma) = \frac{1}{n!} \int_{\mathbb{R}^{nd}} f^{(n)}(y_1, \ldots, y_n) k_\mu^{(n)}(y_1, \ldots, y_n)\, d\sigma(y_1) \cdots d\sigma(y_n)$$

$\mu \rightleftarrows (k_\mu^{(n)})_{n=0}^\infty$

Lenard
K/Kuna
Berezansky/K/Kuna/Lytvynov

Correlation functions dynamics

In components:

$$\frac{\partial k_t^{(n)}}{\partial t} = (L_{CF} k_t)^{(n)}, n \geq 0$$

$$\boxed{\frac{\partial k_t}{\partial t} = L_{CF} k_t}$$

L_{CF}: CF evolution generator, $k_t = (k_t^{(n)})_{n=0}^{\infty}$.

General theory of CF generators for B-A-D processes
and several particular models: [Finkelstein/K/Oliveira '07-'09]

Compare: BBGKY-hierarchy etc.

Harmonic analysis on configuration space

$$\begin{tikzcd}
L:\ \frac{dF_t}{dt}=LF_t & \Gamma,\ F \arrow[rr, leftrightarrow, "\langle F,\mu\rangle=\int_\Gamma F\,d\mu"] \arrow[d, bend left, "K^{-1}"] & & \mathcal{M}^1_{\mathsf{fm}}(\Gamma) \arrow[d, bend right, "K^*"'] & L^*:\ \frac{d\mu_t}{dt}=L^*\mu_t \\
\hat{L}:=K^{-1}LK & \Gamma_0, G \arrow[rr, leftrightarrow, "\langle G,k\rangle=\int_{\Gamma_0} G\,k\,d\lambda"'] \arrow[u, bend left, "K"] & & \mathcal{K}(\Gamma_0) \arrow[u, dotted, bend right, "\text{Lenard}"'] & \hat{L}^*:\ \frac{dk_t}{dt}=\hat{L}^*k_t
\end{tikzcd}$$

K-transform:

$$KG(\gamma):=\sum_{\xi\Subset\gamma}G(\xi),\quad \gamma\in\Gamma.$$

General scheme of construction of non-equilibrium process

- Let $\rho : \Gamma_0 \to \mathbb{R}$ be arbitrary positive function. Denote

$$\mathcal{M}^1_\rho(\Gamma) := \left\{ \mu \in \mathcal{M}^1(\Gamma) \mid k_\mu \le const \cdot \rho \right\}.$$

- Let L be a Markov pre-generator defined on some set of functions $\mathcal{F}(\Gamma)$ given on the configuration space Γ.

$$\frac{\partial F_t}{\partial t} = LF_t, \quad (KE)$$

$$F_t = e^{tL} F_0 \quad (Markov\, semigroup)$$

$$\frac{\partial \mu_t}{\partial t} = L^\star \mu_t, \quad (DKE = FPE)$$

$$\mu_t = e^{tL^\star} \mu_0 \quad (state\, evolution)$$

- Let
$$\hat{L} := K^{-1} L K$$
be a formal K- *transform* or *symbol* of the operator L (our starting object).
- We consider
$$\hat{L} : D(\hat{L}) \subset \mathcal{L} \to \mathcal{L}$$
in a Banach space
$$\mathcal{L} := L^1(\Gamma_0, \rho\, d\lambda_1) = \bigoplus_{n=0}^{\infty} L^1\left(\Gamma^{(n)},\, \rho^{(n)} \sigma^{(n)}\right).$$
Suppose that domain of this operator is such that it is closed and densely defined in $\mathcal{L}$.
- Suppose that $(\hat{L},\, D(\hat{L}))$ is a generator of a semigroup in $\mathcal{L}$:
$$\hat{L} \to \hat{U}_t,\, t \geq 0$$

- Introducing duality between Banach spaces $\mathcal{L}$ and

$$\mathcal{K}(\rho) := \left\{ k : \Gamma_0 \to \mathbb{R} \mid k \cdot \rho^{-1} \in L^\infty(\Gamma_0, \lambda_1) \right\} :$$

$$\langle\!\langle G, k \rangle\!\rangle :=$$

$$= \int_{\Gamma_0} G \cdot k d\lambda_1 = \int_{\Gamma_0} G \cdot \frac{k}{\rho} \cdot \rho d\lambda_1, \; G \in \mathcal{L},$$

  we construct dual evolutional family $\hat{U}_t^\star$, $t \geq 0$ on $\mathcal{K}(\rho)$.

- Suppose that function ρ in the definition of $\mathcal{K}(\rho)$ satisfies Ruelle-type bound. Let $k \in \mathcal{K}(\rho)$ is a correlation function (i.e. the corresponding correlation measure is normalized, positive definite) and let

$$k_t := \hat{U}_t^\star k, \; t \geq 0$$

  denotes an evolution of function k.

- Assume that for any $t \geq 0$, $k_t \in \mathcal{K}(\rho)$ is positive definite, normalized function.

By the main CF theorem one can easily construct an evolution on $\mathcal{M}_\rho^1$:

$$k \quad \rightarrow \quad \mu,$$

$$k_t = \hat{U}_t^\star k \quad \rightarrow \quad U_t^\star \mu, \;\; t \geq 0,$$

$$\mu_t = U_t^\star \mu \in \mathcal{M}_\rho^1.$$

The existence of semigroup $U_t^\star, \, _{t\geq 0}$ on $\mathcal{M}_\rho^1$ implies the existence of process $(X_t^\mu)_{t\geq 0}$ associated with generator L for any initial distribution $\mu \in \mathcal{M}_\rho^1$.

Spatial birth-and-death processes

We consider a Markov pre-generator on Γ, the action of which is given by

$$(LF)(\gamma) := (L_{b,d}F)(\gamma) =$$

$$= \sum_{x\in\gamma} d(x, \gamma \setminus x) D_x^- F(\gamma) + \int_{\mathbb{R}^d} b(x, \gamma) D_x^+ F(\gamma) dx,$$

$D_x^- F(\gamma) = F(\gamma \setminus x) - F(\gamma)$,

$D_x^+ F(\gamma) = F(\gamma \cup x) - F(\gamma)$.

Stochastic growth

BIO: independent growth (plants)

Dispersion kernel:

$a^+(x-y)\,dx$
$0 \le a^+ \in L^1(\mathbb{R}^d)$ even.

Generator:

$$(L_{IG}F)(\gamma) = \sum_{y\in\gamma} \int_{\mathbb{R}^d} a^+(x-y)[F(\gamma\cup x) - F(\gamma)]\,dx$$

Stochastic growth

SOCIO-ECO: migration model (=free development = spontaneous birth = creation by an outer source)

$$(L_{MM}F)(\gamma) = z\int_{\mathbb{R}^d}(F(\gamma\cup x) - F(\gamma))dx$$

GENETICS: generalized mutation models

Evans/Steinsalz/Wichtner, 2005

Density of population

Independent Growth: $k_t(x) \sim Ce^{\lambda t}$, $t \to \infty$

Migration Model: $k_t(x) \sim Ct, t \to \infty$

$$\boxed{Free\ \ Growth\ \ Models}$$

Problem: to analyze stochastic evolution models in presence of global regulations and local competitions

IG with mortality = Contact Model

Generator:

$(LF)(\gamma) = \sum_{x\in\gamma} m[F(\gamma\setminus x) - F(\gamma)]+$

$\sum_{y\in\gamma} \int_{\mathbb{R}^d} a^+(x-y)[F(\gamma\cup x) - F(\gamma)]\,dx$

$m =$ global mortality intensity

Existence of the Markov process:

[K/Skorokhod, 06] finite range a^+

[Finkelstein/K/Skorokhod, 07] general case a^+

Spatial branching with dispersion and mortality (emigration)

Correlation equations for CM

[K/Kutoviy/Pirogov, '08]

What can happen?

Take translation-invariant initial condition

$k_t(x) \equiv \rho_t$.
$\frac{\partial \rho_t}{\partial t} = -(m - \langle a^+ \rangle)\rho_t$
$m > \langle a^+ \rangle \Rightarrow \rho_t \to 0$
$m < \langle a^+ \rangle \Rightarrow \rho_t \to +\infty$
$m = \langle a^+ \rangle \Rightarrow \rho_t \equiv \rho_0$
$m = \langle a^+ \rangle$ critical value of mortality

Possible invariant states: for $m = \langle a^+ \rangle$
that is the critical value $m = m_{\mathsf{cr}}$

CF time evolution

$$\frac{\partial k_t^{(n)}}{\partial t}(x_1, \ldots, x_n) =$$

$$-nmk_t^{(n)}(x_1, \ldots, x_n) +$$

$$\sum_{i=1}^{n} \left[\sum_{j \neq i} a^+(x_i - x_j) \right] k_t^{(n-1)}(x_1, \ldots, \check{x}_i, \ldots, x_n) +$$

$$\sum_{i=1}^{n} \int_{\mathbb{R}^d} a^+(x_i - y) k_t^{(n)}(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n)\, dy$$

Theorem

Let $k_0^{(n)} \leq C^n n!$, $n \geq 0$.
Then $k_t^{(n)}(x_1, \ldots, x_n) \leq A^n (C+t)^n e^{n(<a^+>-m)t} n!$.

Remark:

actually for a Poisson initial state and $x_1, ..., x_n$ inside a small ball

$$k_t^{(n)}(x_1, \ldots, x_n) \simeq C_t^n n!, \;\; t > 0, n \to \infty,$$

that means

STRONG CLUSTERING!

Invariant measures

Let $d \geq 3$.

Stationary equation:

$$\frac{\partial k_t}{\partial t} = 0$$

Theorem

Assume a^+*:* $\int_{\mathbb{R}^d} |x|^2 a^+(x)\, dx < \infty$.

$\forall \rho > 0\ \exists!$ *equilibrium solution* $(k^{(n),\rho})_{n=0}^{\infty}$ *corresponding to a measure*

$\mu^\rho \in \mathcal{M}^1(\Gamma)$
with the density $k^{(1),\rho}(x) = \rho$.
We have

$$k^{(n),\rho}(x) \leq C(\rho)^n (n!)^2, \ \ n \geq 1$$

.

For $d \leq 2$ μ^{ρ} does not exist!

The point:

$$\int_{|p|\leq 1} \frac{dp}{\tilde{a}(0) - \tilde{a}(p)} < \infty$$

necessary condition for the existence of $k^{(2),\rho}(x, y)$.

CM + Kawasaki dynamics (plankton model)

K/Kutoviy/Struckmeier '08
Equilibrium state for CM needs $d \geq 3$.
$d = 2$?
Consider a CM with a motion of individuals:
take into account different time scales
Generator in the bio-time scale:

$$\begin{aligned}(LF)(\gamma) &= \sum_{x\in\gamma}\int_{\mathbb{R}^d} \varkappa(x-y)[F(\gamma\setminus x\cup y)-F(\gamma)]\,dy + (L_{\mathsf{CM}}F)(\gamma)\\ &= ((L_{\mathsf{K}}+L_{\mathsf{CM}})F)(\gamma)\end{aligned}$$

$$\boxed{L = L_{\mathsf{K}}(\varkappa) + L_{\mathsf{CM}}(m, a^+)}$$

Critical value $m = \langle a^+ \rangle$.

Assume heavy tail jumps:

$$\int_{|p|\leq 1} \frac{1}{1 - \tilde{\varkappa}(p)}\, dp < \infty$$

For example:

$\tilde{\varkappa}(p) = e^{-|p|^\alpha}$,

$1 \leq \alpha < 2$ (stable distributions)

Theorem

$\forall \rho > 0 \; \exists!$ *invariant measure* μ^ρ *for CM with Kawasaki dynamics.*

INTERPRETATION: super-diffusive stochastic dynamics of individuals in bio-time scale.

Independent birth with mortality

Surgailis process:

$(LF)(\gamma) = m\sum_{x\in\gamma}(F(\gamma\setminus x) - F(\gamma)) +$
$z\int_{\mathbb{R}^d}(F(\gamma\cup x) - F(\gamma))dx$

Unique invariant measure is the Poisson measure on Γ with intensity

$$\frac{z}{m}$$

Any non-zero mortality stabilizes the system.

BDLMP model

Bolker-Dieckmann-Law-Metz-Pacala model
(= CM + density dependent mortality)

$$L = L_{CM} + L(a^-)$$

$$(LF)(\gamma) = \sum_{x\in\gamma}\left(m + \sum_{x'\in\gamma\setminus x} a^-(x-x')\right)[F(\gamma\setminus x) - F(\gamma)]$$
$$+\sum_{y\in\gamma}\int_{\mathbb{R}^d} a^+(x-y)[F(\gamma\cup x) - F(\gamma)]dx$$

$a^+(x-y)$ – dispersion kernel
$a^-(x-y)$ – competition kernel
$0 \le a^\pm \in L^1(\mathbb{R}^d)$

CF evolution

[Finkelshtein/K/Kutoviy, '09]
Generator on correlation functions:

$$(L_{CF}k)(\eta) = -k(\eta)\left[m|\eta| + E^{a^-}(\eta)\right] -$$

$$\int_{\mathbb{R}^d} \sum_{y\in\eta} a^-(x-y)k(\eta\cup x)dx +$$

$$+\sum_{x\in\eta} k(\eta\setminus x)\sum_{y\in\eta\setminus x} a^+(x-y) +$$

$$\int_{\mathbb{R}^d} \sum_{x\in\eta} a^+(x-y)k(\eta\setminus x\cup y)dy.$$

Ecological problem

May the competition mechanism lead to

THE "REGULARLY DISTRIBUTED" IN SPACE POPULATION

with bounded in time density?

Possible condition: big enough mortality m to control density

Competition assumptions (CA):

$\exists\, C > 0 : \; a^{+}(x) \le C a^{-}(x)$

$m \, > \, C\langle a^{-}\rangle + \langle a^{+}\rangle$

that means strong enough competition and mortality

Theorem

Let (CA) fulfilled. For any initial CF s.t.

$$k_0^{(n)} \leq C^n, n \in \mathbb{N}$$

there exists the unique solution of the CF equation $k_t^{(n)}, t \geq 0, n \in \mathbb{N}$, *satisfying the same bound*

This gives a sub-Poissonian bound,

i.e., strong enough competition

destroys clustering in CM!

CM with establishment

[Finkelshtein/K. 2007]

$a^+(x-y) \ \longrightarrow \ a^+(x-y)e^{-E^{\phi}(x,\gamma)}$

$e^{-E^{\phi}(x,\gamma)} = e(x,\gamma)$ establishment rate

$E^{\phi}(x,\gamma) = \sum_{x'\in\gamma} \phi(x-x')$

competition for survival

Bolker-Pacala establishment rate

$$e(x,\gamma) = \frac{1}{1+E^{\phi}(x,\gamma)}$$

Generator

$$(LF)(\gamma) =$$

$$\sum_{x\in\gamma} m[F(\gamma\setminus x) - F(\gamma)] +$$

$$\sum_{y\in\gamma}\int_{\mathbb{R}^d} a^+(x-y)e(x,\gamma)[F(\gamma\cup x) - F(\gamma)]dx$$

Existence of MP

Stability condition:

$$m > 0, \quad \exists\, \beta > 0 : \ a^+(x) \leq \beta\phi(x)$$

$L = L(m, a^+, \phi)$

Theorem

Under the stability condition there exists a unique Markov process on Γ for $L(m, a^+, \phi)$ (for certain class of initial configurations).

Proof is based on a delicate use of

a [Garcia/Kurtz, 2006] existence result for B-A-D processes

Correlation functions in CM with establishment

Time evolution of CF

Theorem (Finkelshtein/K '07)

Assume stability condition is true and initial CFs satisfy a sub-Poisson bound $k_0^{(n)} \leq C^n, n \geq 1$.
Then there exists $C_1 > 0$ *s.t. for the time dependent CFs holds*

$$k_t^{(n)} \leq C_1^n, n \geq 1.$$

Remark:
The establishment mechanism is more effective
comparing with density dependent mortality:

we do not need to have a big enough mortality
to organize control over time-space behavior
of the population

Density dependent fecundity

$$b(x,\gamma) = \sum_{y\in\gamma} a^{+}(x-y) f(y, \gamma\setminus y)$$

,

$$f(y,\gamma\setminus y) = e^{-E^{\phi}(y,\gamma\setminus y)}$$

OPEN PROBLEM:
PRODUCES THE FECUNDITY AN EFFECTIVE REGULATION OF THE POPULATION?

Dieckmann-Law model

[Finkelshtein/Kondratiev, '09] (in preparation)

$$(LF)(\gamma) = \sum_{x\in\gamma}\left(m + \sum_{x'\in\gamma\setminus x} a^-(x-x')\right)[F(\gamma\setminus x) - F(\gamma)]$$

$$+\sum_{y\in\gamma}\int_{\mathbb{R}^d} a^+(x-y)\left(1+\sum_{y'\in\gamma\setminus y} b(y-y')\right)[F(\gamma\cup x) - F(\gamma)]dx$$

Dieckmann-Law model without competition

i)DLM without local competition $(a^- \equiv 0)$.

Assume existence of corresponding MP $X_t \in \Gamma$.

For $\Lambda \subset \mathbb{R}^d$ bounded put

$$n_t^\Lambda = \mathbb{E}(|X_t \cap \Lambda|)$$

Then $\exists t_0$:

$$n_t^\Lambda \to +\infty$$

$t \to t_0 - 0$ (explosion).

Dieckmann-Law model: non-explosion

ii)DLM: non-explosion via competition $d \geq 2$
Construction of a Lyapunov functional
(cf. [K/Skorokhod, '06]

$$e_\delta(x) = \frac{1}{1+|x|^\delta}, \;\; \delta > d$$

$$\Psi_\delta(x,y) = e_\delta(x) e_\delta(y) \frac{|x-y|+1}{|x-y|} 1\!\!1_{\{x\neq y\}}$$

$$\mathbb{L}_\delta(\gamma) = \langle e_\delta, \gamma \rangle$$

$$\mathbb{E}_\delta(\gamma) = \sum_{\{x,y\}\subset\gamma} \Psi_\delta(x,y)$$

$$\mathbb{V}_\delta(\gamma) = \mathbb{L}_\delta + \mathbb{E}_\delta$$

Assume

$$a^+(x) \le \frac{A}{(1+|x|)^{2\delta}}$$

Introduce $A_\delta = AC(\delta, d) > 0$

Theorem

Assume

$$a^-(x) \ge 2A_\delta b(x)$$

$\Rightarrow \exists C > 0 :$

$$L\mathbb{V}_\delta(\gamma) \le C\mathbb{V}_\delta(\gamma).$$

If, additionally, $m \ge 2A_\delta \langle a^+\rangle \Rightarrow$

$$L\mathbb{V}_\delta(\gamma) \le 0$$

Independent development with competition

$b(x,\gamma) = z > 0$ (independent birth with constant intensity)

(i) $d(x,\gamma \setminus x) = e^{\beta E(x,\gamma\setminus x)} \Longrightarrow$ generator L^{-}

$E(x,\gamma \setminus x) = \sum_{y\in\gamma\setminus x} \phi(x-y)$

(ii) $d(x,\gamma \setminus x) = m + \sum_{y\in\gamma\setminus x} a^{-}(x-y) \Longrightarrow$ generator L_m

(iii) $d(x,\gamma \setminus x) = \sum_{y\in\gamma\setminus x} a^{-}(x-y) \Longrightarrow$ generator L_0

(i) Glauber dynamics G^-

Symmetrizing measures for L^- are

grand canonical Gibbs measures $\mathcal{G}_{gc}(\beta, \phi, z)$

$L^- \longmapsto X_t \in \Gamma$ equilibrium Glauber dynamics
via the Dirichlet form method [K/Lytvynov, '05]

Non-equilibrium GD:

[K/Kutovyi/Minlos,'08]

via CF evolution.

(ii) Surgailis process with competition

$$(L_mF)(\gamma) = \\ \sum_{x\in\gamma}(m+\sum_{y\in\gamma\setminus x}a^-(x-y))(F(\gamma\setminus x)-F(\gamma))+$$

$$z\int(F(\gamma\cup x)-F(\gamma))dx$$

Here we have a sub-Poissonian evolution of CFs.

(iii) Stabilization via competition

[Finkelshtein/K, '09]

Consider (ii) with $m = 0 \Rightarrow$
pure competition mechanism (i.e., without global regulation)

$$(L_0F)(\gamma) = \sum_{x\in\gamma}\sum_{y\in\gamma\setminus x} a^-(x-y)(F(\gamma\setminus x) - F(\gamma)) +$$

$$z\int(F(\gamma\cup x) - F(\gamma))dx$$

Hierarchical equations for CF do not give any a priori information about the density of the system.

There we need the Markov structure (generator!) to obtain corresponding density bound.

Mean density bound

Let $X_t \in \Gamma$ will be MP corresponding to L_0 and an initial measure with bounded density.

Let $\Lambda = B\left(0, R\right)$ be a ball with center at origin and radius $R > 0$.
Introduce the mean density

$$\rho_t^{\Lambda} := \frac{1}{|\Lambda|} \int_{\Lambda} k_t^{(1)}\left(x\right) dx.$$

Our aim is to produce a time bound for this density
and any such volume Λ.

Theorem

Let a^{-} be a continuous function of positive type on $\mathbb{R}^d$.

Then the mean density

$$\rho_t^{\Lambda} := \frac{1}{|\Lambda|} \int_{\Lambda} k_t^{(1)}(x)\, dx$$

is uniformly bounded in $t > 0$ and Λ.

Comments

We use the explicit form of the generator instead of CF equation.
Denote $\gamma_\Lambda = \gamma \cap \Lambda$.
Then

$$L_0|\gamma_\Lambda| \le -\frac{C}{|\Lambda|}|\gamma_\Lambda|^2 + z|\Lambda|$$

That gives easily for $\rho_t = \rho_t^\Lambda$

$$\frac{d}{dt}\rho_t \le z - C\rho_t^2.$$

The proof follows from an analysis of this differential inequality.

Phase space for two-component systems

- $\Gamma^+ = \Gamma^- = \Gamma$
- $\Gamma^2 = \Gamma^+ \times \Gamma^-$
- $\mathcal{B}\left(\Gamma^2\right) = \mathcal{B}\left(\Gamma^+\right) \times \mathcal{B}\left(\Gamma^-\right)$
- $\gamma = (\gamma^+, \gamma^-)$
- Cylinder function: $F(\gamma^+, \gamma^-) = F(\gamma^+ \cap \Lambda, \gamma^- \cap \Lambda)$

Description of model

- $L = L^{+}_{\mathrm{CM}} + L^{-}_{\mathrm{CM}} + L^{+}_{\mathrm{int}}$ — generator
- Here

$$\begin{aligned}\left(L^{+}_{\mathrm{CM}}F\right)\left(\gamma^{+},\gamma^{-}\right) &= \sum_{x\in\gamma^{+}}\left[F\left(\gamma^{+}\setminus x,\gamma^{-}\right)-F\left(\gamma^{+},\gamma^{-}\right)\right] \\ &\quad + \lambda^{+}\int_{\mathbb{R}^d}\left(\sum_{x'\in\gamma^{+}}a^{+}\left(x-x'\right)\right)\left[F\left(\gamma^{+}\cup x,\gamma^{-}\right)-F\left(\gamma^{+},\gamma^{-}\right)\right]dx\end{aligned}$$

$$\begin{aligned}\left(L^{-}_{\mathrm{CM}}F\right)\left(\gamma^{+},\gamma^{-}\right) &= \sum_{y\in\gamma^{-}}\left[F\left(\gamma^{+},\gamma^{-}\setminus y\right)-F\left(\gamma^{+},\gamma^{-}\right)\right] \\ &\quad + \lambda^{-}\int_{\mathbb{R}^d}\left(\sum_{y'\in\gamma^{-}}a^{-}\left(y-y'\right)\right)\left[F\left(\gamma^{+},\gamma^{-}\cup y\right)-F\left(\gamma^{+},\gamma^{-}\right)\right]dy\end{aligned}$$

Description of model

- complementary birth of "$+$"-particles

$$\left(L^{+}_{\mathrm{int}}F\right)\left(\gamma^{+},\gamma^{-}\right) = \lambda\int_{\mathbb{R}^d}\left(\sum_{y\in\gamma^{-}} a\left(x-y\right)\right)\left[F\left(\gamma^{+}\cup x,\gamma^{-}\right)-F\left(\gamma^{+},\gamma^{-}\right)\right]dx$$

- The functions a, a^{+}, a^{-} are positive, even, integrable and normalized. The constants $\lambda, \lambda^{+}, \lambda^{-}$ are positive.

Existence of process

- Let
$$a(x), a^{\pm}(x) \le \frac{A}{(1+|x|)^{\beta}},$$
where $\beta > 2d$, then there exist a massive set of starting points for the process $\gamma \mapsto \gamma_t$
[Finkelshtein/Kondratiev/Skorokhod, '07]
- The problem: studying the behavior of correlation functions for $t \to \infty$
[Filonenko/Finkelshtein/Kondratiev, '08]

Dynamics of correlation functions

- $\Gamma_0^2 = \Gamma_0^+ \times \Gamma_0^-$
- $K^+ := K \otimes 1^-, \quad K^- = 1^+ \otimes K$
- Scheme

$$\begin{tikzcd}
\mathcal{F}(\Gamma^2) \arrow[r, "\langle\cdot,\cdot\rangle"] & \mathcal{M}^1(\Gamma^2) \arrow[d, "(K^+K^-)^*"] \\
\mathcal{F}(\Gamma_0^2) \arrow[u, "K^+K^-"] & \mathcal{M}_\sigma(\Gamma_0^2) \arrow[l, "\langle\cdot,\cdot\rangle"']
\end{tikzcd}$$

- $\hat{L} = (K^+K^-)^{-1} L K^+K^-$
- $\frac{\partial k_t}{\partial t}(\eta^+, \eta^-) = \left(\hat{L}^* k_t\right)(\eta^+, \eta^-)$

Operator $\hat{L}^*$

$$
\begin{aligned}
&\left(\hat{L}^* k\right)\left(\eta^+, \eta^-\right) \\
&= -(\left|\eta^+\right| + \left|\eta^-\right|) k\left(\eta^+, \eta^-\right) \\
&+ \lambda^+ \sum_{x \in \eta^+} \sum_{x' \in \eta^+ \setminus x} a^+(x - x') k\left(\eta^+ \setminus x, \eta^-\right) \\
&+ \lambda^+ \sum_{x \in \eta^+} \int_{\mathbb{R}^d} a^+(x - x') k\left(\eta^+ \setminus x \cup x', \eta^-\right) dx' \\
&+ \lambda^- \sum_{y \in \eta^-} \sum_{y' \in \eta^- \setminus y} a^-(y - y') k\left(\eta^+, \eta^- \setminus y\right) \\
&+ \lambda^- \sum_{y \in \eta^-} \int_{\mathbb{R}^d} a^-(y - y') k\left(\eta^+, \eta^- \setminus y \cup y'\right) dy' \\
&+ \lambda \sum_{x \in \eta^+} \sum_{y \in \eta^-} a(x - y) k\left(\eta^+ \setminus x, \eta^-\right) \\
&+ \lambda \sum_{x \in \eta^+} \int_{\mathbb{R}^d} a(x - y) k\left(\eta^+ \setminus x, \eta^- \cup y\right) dy
\end{aligned}
$$

First-order correlation functions

- Let $k_t^+(x) = k_t(\{x\},\emptyset)$, $k_t^-(y) = k_t(\emptyset,\{y\})$. Then

$$\frac{\partial k_t^-(y)}{\partial t} = -k_t^-(y) + \lambda^- \int\limits_{\mathbb{R}^d} a^-(y-y')k_t^-(y')dy'$$

$$\frac{\partial k_t^+(x)}{\partial t} = -k_t^+(x) + \lambda^+ \int\limits_{\mathbb{R}^d} a^+(x-x')k_t^+(x')dx' + \lambda \int\limits_{\mathbb{R}^d} a(x-y)k_t^-(y)dy$$

- Cauchy problem: initial conditions

$$k_0^-(y) = c^- + \psi^-(y) \geqslant 0, \quad k_0^+(x) = c^+ + \psi^+(x) \geqslant 0$$

where functions $\psi^-,\ \psi^+$ and their Fourier transforms $\hat{\psi}^-,\ \hat{\psi}^+$ are integrable on $\mathbb{R}^d$, and $c^+,\ c^- > 0$.

Second-order correlation functions

$$\begin{aligned}\frac{\partial k_t^{--}(y_1,y_2)}{\partial t} &= -2k_t^{--}(y_1,y_2)\\ &+ \lambda^- a^-(y_1-y_2)[k_t^-(y_1)+k_t^-(y_2)]\\ &+ \lambda^- \int\limits_{\mathbb{R}^d} a^-(y_1-y')k_t^{--}(y_2,y')dy'\\ &+ \lambda^- \int\limits_{\mathbb{R}^d} a^-(y_2-y')k_t^{--}(y_1,y')dy'\end{aligned}$$

Second-order correlation functions

$$\begin{aligned}\frac{\partial k_t^{+-}(x,y)}{\partial t} &= -2k_t^{+-}(x,y)\\ &+ \lambda^+ \int\limits_{\mathbb{R}^d} a^+(x-x')k_t^{+-}(x',y)dx'\\ &+ \lambda^- \int\limits_{\mathbb{R}^d} a^-(y-y')k_t^{+-}(x,y')dy'\\ &+ \lambda a(x-y)k_t^-(y)\\ &+ \lambda \int\limits_{\mathbb{R}^d} a(x-y')k_t^{--}(y,y')dy'\end{aligned}$$

Second-order correlation functions

$$\begin{aligned}\frac{\partial k_t^{++}(x_1,x_2)}{\partial t} &= -2k_t^{++}(x_1,x_2)\\ &+ \lambda^+ \int\limits_{\mathbb{R}^d} a^+(x_1-x')k_t^{++}(x_2,x')dx'\\ &+ \lambda^+ \int\limits_{\mathbb{R}^d} a^+(x_2-x')k_t^{++}(x_1,x')dx'\\ &+ \lambda^+ a^+(x_1-x_2)[k_t^+(x_1)+k_t^+(x_2)]\\ &+ \lambda \int\limits_{\mathbb{R}^d} a(x_1-y)k_t^{+-}(x_2,y)dy\\ &+ \lambda \int\limits_{\mathbb{R}^d} a(x_2-y)k_t^{+-}(x_1,y)dy\end{aligned}$$

Second-order correlation functions

- Cauchy problem:

$$\begin{gathered} k_0^{--}(y_1,y_2) = c^{--} + \varphi^{--}(y_1-y_2) \geqslant 0 \\ k_0^{+-}(x,y) = c^{+-} + \varphi^{+-}(x-y) \geqslant 0 \\ k_0^{++}(x_1,x_2) = c^{++} + \varphi^{++}(x_1-x_2) \geqslant 0 \end{gathered}$$

where $c^{--},\ c^{+-},\ c^{++} > 0$ and $\varphi^{--},\ \varphi^{+-},\ \varphi^{++}$ are integrable functions on $\mathbb{R}^d$ together with their Fourier transforms $\hat{\varphi}^{--},\ \hat{\varphi}^{+-},\ \hat{\varphi}^{++}$.

Asymptotic for densities

Theorem

One has at $t \to \infty$:

$$k_t^+(x) \to \begin{cases} 0, \text{ if } \max\{\lambda^+, \lambda^-\} < 1 \\ \infty, \text{ if } \max\{\lambda^+, \lambda^-\} > 1 \text{ or } \lambda^+ = \lambda^- = 1 \end{cases},$$

and in the case $1 = \lambda^+ > \lambda^-$

$$k_t^+(x) \to c^+ + \frac{\lambda c^-}{1 - \lambda^-},$$

and in the case $\lambda^+ < \lambda^- = 1$

$$k_t^+(x) \to \frac{\lambda c^-}{1 - \lambda^+}$$

The limits for second-order corr. functions

Theorem

1. If $\lambda^+ = 1$, $0 < \lambda^- < 1$ *then*

$$\begin{cases} \lim_{t\to\infty} k_t^{--}(y_1, y_2) = 0 \\ \lim_{t\to\infty} k_t^{+-}(x, y) = 0 \end{cases}$$

and

$$\lim_{t\to\infty} k_t^{++}(x_1, x_2) = \left(c^{++} - \frac{2\lambda c^{+-}}{\lambda^- - 1} + \frac{\lambda^2 c^{--}}{(\lambda^- - 1)^2} \right) + \Omega^{++}(x_1 - x_2) < \infty,$$

where function Ω^{++} *depends only on initial value* c^+ *(and on parameters* $\lambda, \lambda^\pm, a, a^\pm$*).*

The limits for second-order corr. functions

Theorem

2. If $\lambda^- = 1$, $0 < \lambda^+ < 1$ then

$$\begin{cases} \lim_{t\to\infty} k_t^{--}(y_1, y_2) = c^{--} + \Xi^{--}(y_1 - y_2) < \infty \\ \lim_{t\to\infty} k_t^{+-}(x, y) = \frac{\lambda c^{--}}{1-\lambda^+} + \Xi^{+-}(x - y) < \infty \\ \lim_{t\to\infty} k_t^{++}(x_1, x_2) = \frac{\lambda^2 c^{--}}{(1-\lambda^+)^2} + \Xi^{++}(x_1 - x_2) < \infty \end{cases}$$

where functions $\Xi^{--}, \Xi^{+-}, \Xi^{++}$ depend only on initial value c^- (and on $\lambda, \lambda^\pm, a, a^\pm$).

Ursell functions

- Let $\psi^{\pm} = 0$ and $c^{++} = (c^{+})^2$, $c^{+-} = c^{+}c^{-}$, $c^{--} = (c^{-})^2$, then
- for the case $\lambda^{+} = 1$, $0 < \lambda^{-} < 1$

$$\lim_{t\to\infty} \left(k_t^{++}(x_1, x_2) - (k_t^{+})^2\right) = \Omega^{++}(x_1 - x_2)$$

- for the case $\lambda^{-} = 1$, $0 < \lambda^{+} < 1$

$$\begin{cases} \lim_{t\to\infty} \left(k_t^{--}(y_1, y_2) - (k_t^{-})^2\right) = \Xi^{--}(y_1 - y_2) \\ \lim_{t\to\infty} \left(k_t^{+-}(x, y) - k_t^{+}k_t^{-}\right) = \Xi^{+-}(x - y) \\ \lim_{t\to\infty} \left(k_t^{++}(x_1, x_2) - (k_t^{+})^2\right) = \Xi^{++}(x_1 - x_2) \end{cases}$$

- And the functions Ω and Ξ are solutions of equation corresponding to stationary equation

$$\hat{L}^{*}k = 0$$

Description of symbiotic model

- $L_s = L^+_{\mathrm{CM}} + L^-_{\mathrm{CM}} + L^+_{\mathrm{int}} + L^-_{\mathrm{int}} = L + L^-_{\mathrm{int}}$
- Here

$$\begin{aligned}\left(L^-_{\mathrm{int}} F\right)\left(\gamma^+,\gamma^-\right) \\ &= \lambda' \int_{\mathbb{R}^d} \left(\sum_{x\in\gamma^+} a'\left(x-y\right)\right)\left[F\left(\gamma^+,\gamma^-\cup y\right) - F\left(\gamma^+,\gamma^-\right)\right] dy,\end{aligned}$$

- where a' has the same properties as a and $\lambda' > 0$.

First-order correlation functions

- One has

$$\frac{\partial k_t^-(y)}{\partial t} = -k_t^-(y) + \lambda^- \int\limits_{\mathbb{R}^d} a^-(y-y')k_t^-(y')dy' + \lambda' \int\limits_{\mathbb{R}^d} a'(x-y)k_t^+(x)dx$$

$$\frac{\partial k_t^+(x)}{\partial t} = -k_t^+(x) + \lambda^+ \int\limits_{\mathbb{R}^d} a^+(x-x')k_t^+(x')dx' + \lambda \int\limits_{\mathbb{R}^d} a(x-y)k_t^-(y)dy$$

Solution

- Let $k_0^{\pm} = c^{\pm} > 0$. Then $(k_t^+, k_t^-)^{\perp} = e^{tA}(c^+, c^-)^{\perp}$, where

$$\begin{pmatrix} \lambda^+ - 1 & \lambda \\ \lambda' & \lambda^- - 1 \end{pmatrix}$$

- Characteristic equation has positive discriminant (since $\lambda\lambda' > 0$) and, therefore,

$$k_t^{\pm} = C_1^{\pm} e^{\chi_1 t} + C_2^{\pm} e^{\chi_2 t}.$$

- As a result, the first-order correlation function goes to (non-zero) constant only if $\chi_1 < \chi_2 = 0$.
- Sufficient condition:

$$\lambda^{\pm} < 1, \qquad (\lambda^+ - 1)(\lambda^- - 1) = \lambda\lambda'$$

and then $\frac{\lim k_t^+}{\lim k_t^-} = \frac{\lambda}{1-\lambda^+}$.

Heterogeneous Contact Models: random mortality

Lattice case: Joo/Lebowitz, Phys.Rev.E72, 2005

Mortality rate $m \to m(x, \omega) \geq 0$
Density evolution

$$\frac{\partial k_t(x)}{\partial t} = L^{a^+} k_t(x) - V(x) k_t(x)$$

where

$$L^{a^+} f(x) = \int a^+(x-y)[f(y) - f(x)]dy$$

$$V(x, \omega) = m(x, \omega) - \langle a^+ \rangle$$

Parabolic Anderson problem for pure jump generator
(CTRW in continuum)

Random establishment

$$a^{+}(x-y) \to a^{+}(x-y)b(x,\omega)$$

Generator

$$(LF)(\gamma) = \sum_{x\in\gamma} m(x)[F(\gamma\setminus x) - F(\gamma)] +$$

$$\sum_{y\in\gamma}\int_{\mathbb{R}^d} a^{+}(x-y)b(x,\omega)[F(\gamma\cup x) - F(\gamma)]dx$$

Density evolution

$$\frac{\partial k_t(x)}{\partial t} = L_b^{a^+} k_t(x) - V(x) k_t(x)$$

$$L_b^{a^+} f(x) = b(x,\omega) \int a^+(x-y)[f(y) - f(x)]dy$$

$$V(x,\omega) = m(x,\omega) - b(x,\omega)\langle a^+ \rangle$$

$L_b^{a^+}$ is symmetric in

$$L^2(\mathbb{R}^d, b^{-1}(x)dx)$$

(quenched random measure?).

Random fecundity

Random fecundity rate

$$\mathbb{R}^d \ni y \mapsto \varkappa(y,\omega) \geq 0$$

Birth rate

$$b(x,\gamma,\omega) = \sum_{y\in\gamma} a^+(x-y)\varkappa(y,\omega)$$

Density evolution

$$\frac{\partial k_t(x)}{\partial t} = L_{\varkappa}^{a^+} k_t(x) - V(x)k_t(x)$$

$$L_{\varkappa}^{a^+} f(x) = \int a^+(x-y)\varkappa(y,\omega)[f(y) - f(x)]dy$$

$$V(x,\omega) = m(x,\omega) - \langle a^+(x-\cdot)\varkappa(\cdot,\omega)\rangle$$

$L_{\varkappa}^{a^+}$ is symmetric in

$$L^2(\mathbb{R}^d, \varkappa(x,\omega)dx)$$

CTRW in random environment

Two types of quenched jump generators:

$$L_b^{a^+} f(x) = b(x,\omega) \int a^+(x-y)[f(y)-f(x)]dy$$

$$L_\varkappa^{a^+} f(x) = \int a^+(x-y)\varkappa(y,\omega)[f(y)-f(x)]dy$$

Feynman-Kac formula

Quenched CTRW:

$$L_{b,\varkappa}^{a^+,\omega} \to \xi_t^\omega$$

Density of population:

$$k_t(x) = \mathbb{E}_x[k_0(\xi_t^\omega)e^{-\int_0^t V(\xi_s^\omega)ds}]$$

Quenched vs. annealed

(cf., e.g., Donsker/Varadhan, Gaertner/Molchanov, Sznitman, ...)

Other aspects of IBM

– multi-type systems
(Finkelshtein/K; F/K/Skorokhod),

– mutation-selection models in genetics
(K/Minlos/Pirogov; K/Kuna/Ohlerich)

– scaling limits
(Finkelshtein/K/Kuna/Kutovyi; Finkelshtein/K/Lytvynov;
Finkelshtein/K/Kutovyi)

– Kawasaki dynamics in continuum
(K/Lytvynov/Roeckner;K/Kuna/Oliveira/Streit)

– plankton dynamics
(K/Kutovyi/Struckmeier)

– stochastic evolutions in evolving random environments
(Boldrighini/K/Minlos/Pellegrinotti; Struckmeier)

– spectral analysis of Markov generators
(K/Lytvynov; K/Minlos; K/Zhizhina; K/Kuna/Ohlerich)