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Intermittency of higher-order moments in random environments

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1. Ecological populations as complex systems

Populations of various living organisms: very large complex systems.

Spatio-temporal dynamics involves *uncertainty*:

- reproduction and mortality
- competition (establishment/vegetation, food chains/webs)
- spatial mobility (dispersion, foraging, migration)
- genetic variability (selection, adaptation)

Ecosystems are *non-closed*:

- interaction within/between species
- interaction with varying environment

2. Random media theory

Random dynamics in random environments

e.g., branching diffusions with *random* birth/death rates

Two competing mechanisms:

- self-averaging (due to diffusion, dissipation, etc.)
- self-organization (via interaction, branching, local reactions, etc.)

Conventional approach *(Averaging Principle)*:

Replace the random medium by an “effective” (non-random) one

Homogenization theory (classical physics)

“Bulk” limit theorems (laws of large numbers, Gaussian limits)

Large rare fluctuations of the environment may lead to anomalous properties: intermittency, localization, non-classical diffusion . . .

3. Intermittency phenomenon

“Intermittency” = highly irregular spatio-temporal structures:

Hierarchy of high rare peaks on a low-profile background

Typical of many complex processes and systems:

- chemical physics (“strong reaction centres”)
- hydrodynamics (hierarchy of vortices in turbulence)
- astrophysics (formation of galaxies, stars and planets)
- finance, trade and industry (accumulation and redistribution of wealth and resources)
- ecology (population density, competition, migration of species)

Relevant to population dynamics! (speciation, spatial patterns)

Mathematical theory of intermittency in random media

[Ya. B. Zeldovich, S. Molchanov, J. Gärtner, R. Carmona, . . .]

Asymptotic analysis of statistical moments

4. Particle dynamics

Birth and death process

$$\begin{tikzcd} & A + A \\ A \arrow[ur] \arrow[dr] & \\ & \varnothing \end{tikzcd}$$

Spatial motion: diffusion of particles

Branching rates: random, variable in space

Problem: Long-term behaviour of the particle dynamics, e.g.

- population growth
- extinction
- spatial propagation
- formation of spatial patterns (e.g. agglomeration)
- etc.

Simple model: **Branching Random Walk**

Diffusion: random walk X_t on $\mathbb{Z}^d$ $(d \geq 1)$ with *jump rate* $\kappa > 0$

Branching: random *birth* and *death* rates at points $x \in \mathbb{Z}^d$

$$\xi_2(x) = \xi_2(x,\omega) \geq 0 \qquad (1 \rightsquigarrow 2)$$

$$\xi_0(x) = \xi_0(x,\omega) \geq 0 \qquad (1 \rightsquigarrow 0)$$

$\omega \in \Omega$:= sample realization of the random environment $\xi_2,\ \xi_0$

For a fixed $\omega \in \Omega$ a particle at site $x \in \mathbb{Z}^d$ during a small time $h \to 0$ may experience the following (independently of other particles):

Transition	Probability
jump to an adjacent site	$\frac{\kappa}{2d}\,h + o(h)$
or split into two	$\xi_2(x)\,h + o(h)$
or die	$\xi_0(x)\,h + o(h)$
or else stay intact	$1 - \sum(\dots)\,h + o(h)$

5. Mean population size

Let

$$\mu_t(y) := \text{number of particles in } y \in \mathbb{Z}^d \text{ at time } t$$

$$\mu_t := \sum_{y \in \mathbb{Z}^d} \mu_t(y) = \text{population size at time } t$$

Initial condition: Start with one particle, located in $x \in \mathbb{Z}^d$:

$$\mu_0 = 1, \qquad \mu_0(y) = \delta_x(y).$$

Fix $\omega \in \Omega$ (a realization of the branching environment) and consider *quenched* statistical moments

$$m_1(t, x, y) := E_x^\omega \, \mu_t(y), \qquad m_1(t, x) := E_x^\omega \, \mu_t$$

$$x = \text{position of single ancestor at } t = 0$$

$$E_x^\omega := \text{expectation in a fixed environment } \omega$$

Backward Kolmogorov equation for $m_1 = m_1(t,x)$:

Decompose with respect to transitions during initial time $h \to 0$:

$$
\begin{aligned}
m_1(t+h,x) = \frac{\kappa h}{2d} \sum_{x'} m_1(t,x') & \qquad [\, x' : |x'-x| = 1\,] \\
+ 2\, m_1(t,x)\, \xi_2(x)\, h & \\
+ \Big[1 - \kappa h - \xi_2(x) h - \xi_0(x) h\Big] m_1(t,x) + o(h), &
\end{aligned}
$$

whence

$$
\boxed{\frac{\partial m_1}{\partial t} = \kappa \Delta m_1 + \xi(x) m_1, \qquad m_1(0,x) \equiv 1} \qquad (*)
$$

Here

$$
\Delta \psi(x) := \frac{1}{2d} \sum_{x'} \psi(x') - \psi(x) \qquad \text{(lattice Laplacian)}
$$

$$
\xi(x) := \xi_2(x) - \xi_0(x) \qquad \text{(random potential)}
$$

Note that

$$\boxed{\frac{\partial m_1}{\partial t} = \kappa \Delta m_1 + \xi(x) m_1, \qquad m_1(0,x) \equiv 1} \qquad (*)$$

is a Cauchy problem for the *Anderson operator* $H = \kappa\Delta + \xi(x)$

Similarly, local moments $m_1 = m_1(t,x,y)$ satisfy

$$\boxed{\frac{\partial m_1}{\partial t} = \kappa \Delta m_1 + \xi(x) m_1, \quad m_1(0,x,y) \equiv \delta_y(x)}$$

Let $\mathbb{P} :=$ distribution of the environment $\xi_2(\cdot),\ \xi_0(\cdot)$

$\langle \cdot \rangle :=$ expectation with respect to $\mathbb{P}$

6. Lyapunov exponents for $m_1(t,x)$

Annealed moments: $\langle m_1(t,x)^p \rangle \quad (p \in \mathbb{N})$

Theorem [Gärtner & Molchanov, 1990]

Assume that $\xi(x) \ge 0$ *are i.i.d. with Weibull tail*:

$$\mathbb{P}\{\xi(x) > y\} \sim e^{-y^\alpha}, \qquad y \to \infty \qquad (\alpha > 1)$$

Put $\alpha' := \dfrac{\alpha}{\alpha - 1} > 1$. *Then*

$$\lambda_p := \lim_{t\to\infty} \frac{\log \langle m_1(t,x)^p \rangle}{t^{\alpha'}} = c_d(\alpha)\, p^{\alpha'} \qquad (p = 1, 2, \dots)$$

(*annealed Lyapunov exponents*),

$$\tilde\lambda := \lim_{t\to\infty} \frac{\log m_1(t,x)}{t\,(\log t)^{1/\alpha}} = d^{1/\alpha} \qquad (\mathbb{P}\text{-a.s.})$$

(*quenched Lyapunov exponent*).

That is, $\boxed{\langle m_1(t,x)^p \rangle \approx e^{\lambda_p t^{\alpha'}}, \qquad m_1(t,x) \approx e^{\tilde\lambda\, t\,(\log t)^{1/\alpha}}}$

Feynman–Kac formula

Define

$$\log_+ x := \begin{cases} \log x, & x \geq e, \\ 1 & x < e. \end{cases}$$

Theorem [Gärtner & Molchanov, 1990]

Assume that $\xi(x) \geq 0$ *are i.i.d. Then the Cauchy problem* $(*)$ *has a unique non-negative solution if and only if*

$$\left\langle \left(\frac{\xi(0)}{\log_+ \xi(0)} \right)^d \right\rangle < \infty.$$

This solution admits the Feynman–Kac representation

$$m_1(t,x) = E_x\Big[e^{\int_0^t \xi(X_s)\,ds}\Big],$$

where E_x = *expectation with respect to a random walk* X_t $(X_0 = x)$ (*nearest jumps with rate* κ, *no branching*).

Proof of the moment part:

For a Weibull $\xi(0)$ we have

$$G(t) := \log \left\langle e^{t\xi(0)} \right\rangle \sim c_d(\alpha)\, t^{\alpha'} \quad (t \to \infty)$$

Lower bound: In the Feynman-Kac formula, set $X_s \equiv x$ $(0 \leq s \leq t)$, which has probability $e^{-\kappa t}$:

$$m_1(t,x) \geq e^{-\kappa t} \cdot e^{\xi(x)t}, \quad \text{hence}$$

$$\langle m_1(t,x)^p \rangle \geq e^{-p\kappa t} \langle e^{pt\xi(x)} \rangle = e^{-p\kappa t + G(pt)}.$$

Upper bound:

$$\begin{aligned}
\langle m_1(t,x)^p \rangle &= \left\langle \left(E_x \left[\exp\left\{ \int_0^t \xi(X_s)\, ds \right\} \right] \right)^p \right\rangle && \text{(Feynman-Kac formula)} \\
&\leq \left\langle E_x \left[\exp\left\{ p \int_0^t \xi(X_s)\, ds \right\} \right] \right\rangle && \text{(Jensen's inequality)} \\
&\leq E_x \left\langle \frac{1}{t} \int_0^t e^{pt\xi(X_s)}\, ds \right\rangle && \text{(Jensen's inequality)} \\
&= \frac{1}{t} \int_0^t E_x \left\langle e^{pt\xi(X_s)} \right\rangle ds && \text{(Fubini's theorem)} \\
&= \langle e^{pt\xi(0)} \rangle = e^{G(pt)} && \text{($\xi(x)$ are i.i.d.)}
\end{aligned}$$

7. Intermittency of $m_1(t,x)$

Note:

$$\frac{1}{p}\lambda_p = c_d(\alpha)p^{\alpha'-1} \nearrow\nearrow \text{ in } p \qquad (\text{because } \alpha' > 1)$$

That is,

$$\boxed{\lambda_1 < \tfrac{1}{2}\lambda_2 < \tfrac{1}{3}\lambda_3 < \dots}$$

$$\Rightarrow \langle m_1^2\rangle \gg \langle m_1\rangle^2, \quad \langle m_1^3\rangle \gg \langle m_1^2\rangle^{3/2}, \quad \dots$$

progressive growth of moments

This implies **intermittency** of $m_1(t,x)$ as $t \to \infty$

Explanation:

Pick $r: \ \lambda_1 < r < \frac{1}{2}\lambda_2$ and consider (random) sets

$$B_r(t) := \{x \in \mathbb{Z}^d : \ m_1(t,x) > e^{rt^{\alpha'}}\}$$

By Chebyshev's inequality,

$$\mathbb{P}\{x \in B_r(t)\} \le \langle m_1(t,x)\rangle \cdot e^{-rt^{\alpha'}}$$
$$\approx e^{-(r-\lambda_1)t^{\alpha'}} \to 0 \qquad (t \to \infty).$$

hence the spatial density of $B_r(t)$ is exponentially small. But

$$\left\langle m_1^2(t,x) \cdot \mathbf{1}_{\{x \notin B_r(t)\}} \right\rangle \le e^{2rt^{\alpha'}}$$
$$\ll e^{\lambda_2 t^{\alpha'}} \approx \langle m_1^2(t,x)\rangle,$$

i.e., $\langle m_1^2 \rangle$ is basically formed on the set $B_r(t)$!

8. Higher-order moments

Moments $m_n(t,x) := E_x^{\omega}\mu_t^n$ satisfy the chain of evolution equations

$$\frac{\partial m_n}{\partial t} = \kappa\Delta m_n + \xi(x)\,m_n + \xi_2(x)\,f_n,$$

$$m_n(0,x) \equiv 1, \quad n = 1, 2, \ldots$$

$$f_n = \sum_{i=1}^{n-1} \binom{n}{i} m_i\, m_{n-i}$$

Convenient to use the quenched Laplace transform

$$F(z;t,x) := E_x^{\omega}\, e^{-z\mu_t} \quad (z \ge 0) \quad \Rightarrow \quad m_n = (-1)^n \frac{\partial F(z;t,x)}{\partial z}\Big|_{z=0}$$

which satisfies

$$\frac{\partial F}{\partial t} = \kappa\Delta F + \Big(\xi_2(x)F - \xi_0(x)\Big)\,(F-1), \qquad F(z;0,x) = e^{-z}$$

Theorem [Albeverio, Bogachev, Molchanov & Yarovaya, 2000]

Same assumptions as above (*i.i.d.* $\xi(x)$, *Weibull tail with* $\alpha > 1$). *Then*

$$\lambda_{n,p} := \lim_{t\to\infty} \frac{\log \langle m_n(t,x)^p\rangle}{t^{\alpha'}} = c_d(\alpha)\,(np)^{\alpha'},$$

$$\alpha' := \frac{\alpha}{\alpha-1}, \quad n = 1,2,\ldots, \quad p = 1,2,\ldots$$

Feynman–Kac type representation:

$$\begin{aligned} m_n(t,x) &= E_x\Big[e^{\int_0^t \xi(X_s)\,ds}\Big] \\ &\quad + E_x\Big[\int_0^t e^{\int_0^t \xi(X_u)\,du}\,\xi_2(X_s)\,f_n(t-s,X_s)\,ds\Big] \end{aligned}$$

9. Adequacy of moments?

Recall: $\lambda_{n,p} = c_d(\alpha)\,(np)^{\alpha'}, \qquad \lambda_{1,p} = c_d(\alpha)\,p^{\alpha'},$

hence

$$\boxed{\lambda_{n,p} = \lambda_{1,np}} \quad \Rightarrow \quad \langle m_n^p \rangle \approx \langle m_1^{np} \rangle \text{ !}$$

That is, relative to m_1, higher-order moments m_n grow in a regular way, and so intermittency of m_n (and of μ_t) is “reduced” to that of m_1.

This is supported by quenched Lyapunov exponents [Molchanov, 1996]

$$\tilde{\lambda}_n := \lim_{t\to\infty} \frac{\log m_n(t,x)}{t\,(\log t)^{1/\alpha}} = n d^{1/\alpha} \quad (\mathbb{P}\text{-a.s.})$$

whence

$$\boxed{\tilde{\lambda}_n = n\tilde{\lambda}_1} \quad \Rightarrow \quad m_n \approx m_1^n \quad (\mathbb{P}\text{-a.s.}) \text{ !}$$

Particle process μ_t is governed by the "annealed" probability measure

$$\mathcal{P}(\cdot) = \langle P_x^\omega(\cdot) \rangle$$

Conjecture (Molchanov, Bogachev):

Adequacy Principle. *The ratio* $\rho(t) := \dfrac{\mu_t}{m_1(t, x)}$ *is bounded above and below in* $\mathcal{P}$*-probability, i.e. uniformly in* $t \geq 0$, *for any* $\epsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\mathcal{P}\{\delta \leq \rho(t) \leq \delta^{-1}\} \geq 1 - \epsilon.$$

This would imply

$$\frac{\log \mu_t}{\log m_1(t, x)} \xrightarrow{\mathcal{P}} 1, \qquad t \to \infty,$$

provided that

$$m_1(t, x) \xrightarrow{\mathbb{P}} \infty, \qquad t \to \infty.$$

(Still open. . .)

10. Conclusion

Some generalizations and extensions:

- Non-static environment $\xi(t,x) = \dot{B}(t,x)$
  [Carmona & Molchanov, 1994]
- Continuous space
  [Carmona & Molchanov, 1995], [Gärtner & König, 2000],
  [Gärtner, König & Molchanov, 2000]
- Geometric structure of intermittency
  [Gärtner & Molchanov, 1998], [Gärtner, König & Molchanov, 2007]
- Four universality classes (distribution of $\xi(x)$)
  [Hofstad, König & Mörters, 2006]
- Spatio-temporal scales
  [Ben Arous, Molchanov & Ramírez, 2007]
  see also [Ben Arous, Bogachev & Molchanov, 2005]
- $\ldots$

References

[1] Ya. B. Zeldovich, S. A. Molchanov, A. A. Ruzmaikin & D. D. Sokoloff. Intermittency, diffusion and generation in a nonstationary random medium. *Sov. Sci. Rev., Sect. C: Math. Phys. Rev.* **7** (1988), 1–110.

[4] J. Gärtner & S. A. Molchanov. Parabolic problems for the Anderson model. I: Intermittency and related topics. *Comm. Math. Phys.* **132** (1990), 613–655.

[2] S. A. Molchanov. Lectures on random media. *Lect. Notes Math.* **1581** (1994), 242–411.

[3] R. Carmona & S. A. Molchanov. *Parabolic Anderson Problem and Intermittency*. Amer. Math. Soc., Providence, RI, 1994.

[5] R. Carmona & S. A. Molchanov. Stationary parabolic Anderson model and intermittency. *Probab. Theory Relat. Fields* **102** (1995), 433–453.

[6] S. A. Molchanov. Reaction-diffusion equations in the random media: Localization and intermittency. In: *Nonlinear Stochastic PDEs. Hydrodynamic Limit and Burgers' Turbulence* (T. Funaki et al., eds.). IMA, Math. Appl. **77** (1996), 81–109. Springer, Berlin.

[5] J. Gärtner & S. A. Molchanov. Parabolic problems for the Anderson model. II: Second-order asymptotics and structure of high peacks. *Probab. Theory Relat. Fields* **111** (1998), 17–55.

[7] S. Albeverio, L.V. Bogachev, S. A. Molchanov & E. B. Yarovaya. Annealed moment Lyapunov exponents for a branching random walk in a homogeneous random branching environment. *Markov Proc. Relat. Fields* **6** (2000), 473–516.

[8] J. Gärtner & W. König. Moment asymptotics for the continuous parabolic Anderson model. *Ann. Appl. Probab.* **10** (2000), 192–217.

[9] J. Gärtner, W. König & S. Molchanov. Almost sure asymptotics for the continuous parabolic Anderson model. *Probab. Theory Related Fields* **118** (2000), 547-573.

[10] G. Ben Arous, L.V. Bogachev & S. A. Molchanov. Limit theorems for sums of random exponentials. *Probab. Theory Related Fields* **132** (2005), 579–612.

[11] R. van der Hofstad, W. König & P. Mörters. The universality classes in the parabolic Anderson model. *Comm. Math. Phys.* **267** (2006), 307-353.

[12] J. Gärtner, W. König & S. Molchanov. Geometric characterization of intermittency in the parabolic Anderson model. *Ann. Probab.* **35** (2007), 439-499.

[13] G. Ben Arous, S. Molchanov & A. F. Ramírez. Transition asymptotics for reaction-diffusion in random media. In: *Probability and Mathematical Physics: A Volume in Honor of Stanislav Molchanov* (D. A. Dawson et al., eds.), CRM Proc. Lect. Notes **42**, 1–40. Amer. Math. Society, Providence, RI, 2007.