

Report on Workshop: The higher-genus sigma function and applications
Organisers: C. Athorne, H. Braden, V. Buchstaber, J. C. Eilbeck (Local Organiser), V. Z. Enolski, J. Gibbons, & E. Previato.

Relation to original proposal

The main goals of the meeting were fulfilled, although not all the originally scheduled speakers were able to attend. This was partly due to a delay between the original grant submission and the confirmation of funding, and partly to some technical email problems which have now been fixed.

Overview

The meeting was devoted to exploring areas of algebraic curves and their moduli spaces; transcendental (special) functions; and their applications in applied mathematics and mathematical physics. These areas are linked by higher genus generalisations of the Weierstrass sigma function and their logarithmic derivatives which are generalized Weierstrass \wp functions. Recently this area has been revitalised by the interaction of “experimental” results from computer algebra with theoretical developments, and the workshop brought together some leading workers in this area. In classical complex analysis, the Weierstrass σ -function plays a central role because it captures both the algebraic (addition rule) and the transcendental properties of elliptic curves, with consequent applications.

Report

The area of algebraic curves and their moduli spaces and related special functions has recently been revived by new techniques and the increasing power and functionality of computer algebra software. This has spurred a number of interesting developments on the theoretical side, and also a number of novel applications. The workshop brought together practitioners in all these areas for an intensive week long meeting in South College Street. The feedback on the scientific content of the meeting and the new facilities at ICMS was almost universally positive.

Notes or presentations of all the talks given at the workshop, are available at <http://www.ma.hw.ac.uk/~chris/icms/Sigma/>.

On the theory side, Nakayashiki discussed a deformation of Baker’s addition formula for sigma functions, and Ônishi talked about Frobenius-Stickelberger addition formulae for general curves. Grant reported work on modular models in genus 2, and Previato explored specific sub-varieties of moduli spaces of curves. Korotkin explained a new way of viewing the isomonodromic τ -function as a higher genus analogue of the Dedekind η -function. Treibich reported his recent work on smooth hyperelliptic covers and systems of polynomial equations. Eilbeck talked about the connection of sigma functions with tau functions, and also reported some work on explicit equations for Kummer varieties in genus 3 and 4. The theme of higher genus Abelian functions was continued by England, and by Athorne.

On the applications side there were a number of talks on the use of sigma functions to solve various problems in mathematical physics, fluid mechanics, and other areas. Gibbons discussed the use of sigma functions to find solutions to the Benney equation arising in fluid mechanics. In the talks of Enolski and Braden, the Hitchin theory of non-abelian monopoles was discussed from the viewpoint of multi-variable theta-functions, involving delicate theta-functional and number-theoretical results such as the Ramanujan hypergeometric relations. Schottky-Jung proportionalities were applied to construct new monopole solutions to the Bogomolny equation. Kunz discussed multi-monopoles and monopole-antimonopole systems within Georgi-Glashow model. Most of these solutions are known only numerically, which presents a challenge for future applications of algebro-geometric methods. In a similar area, Kalla discussed algebro-geometric solutions of the Davey-Stewartson equation and the vector nonlinear Schrödinger equation.

A number of members of the Bremen group (Lämmerzahl, Hartmann, Kagramanova) gave talks on applications to relativity theory and cosmology. The talk by Lämmerzahl described the

complete set of analytic solutions of the geodesic equation in the class of Plebanski-Demianski space-times, and demonstrated effective use of multi-variable sigma-functions to describe observable effects like the Perihelion shift, the Lense-Thirring effect, the deflection of light, and others. Kagramanova's talk was devoted to the description of motion of a test particle in various metrics, resolving the Einstein equation in terms of inversion of higher genera hyperelliptic integrals. Hartmann considered cosmic strings and the relevance of analytic methods that can be used for their description in the framework of sigma function theory. Ritzenthaler gave an overview of applications of sigma function theory to algorithmic number theory and coding theory, and cryptography, and Hone discussed analytic solutions and integrability for bilinear recurrences. A poster by Swierczewski described how bi-tangents of quartics could be computed using Riemann Theta Functions.

Talks in related areas were given by Mironov (commuting ordinary differential operators of rank 2), Shpectorov (computing braid orbits), and Fedorov (Neumann systems on Stiefel varieties and matrix Jacobi-Mumford systems).

On the computational side, new impetus was given especially to seeking explicit solutions of Benney equations, magnetic monopoles and the Schrödinger equation in one and several variables. On the theoretical side, experts from nine different countries were excited to compare ideas for innovative uses of the sigma function in the geometry of moduli spaces and related constructions. At the crossroads, communication between the computer-system developers and the theoreticians motivated promising work on fast implementation of the calculations of Thetanullwerte, period matrices, Thomae's formulae, e.g., which would be applied to classical constructions (for example monodromy equations over moduli spaces, Schottky-type equations, the Arithmetic-geometric Mean) to give new results in algebraic and differential geometry, coding theory and cryptography.

We are grateful to ICMS for making this meeting possible and to all the staff for their help in making everything go smoothly.