Non-linear stability for radiative spacetimes: Vacuum and Einstein-Maxwell

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arXiv:0910.4073 [vacuum]

Overview

- **Aim:** Analyse the stability near future timelike infinity i^+ for radiative vacuum and Einstein-Maxwell spacetimes $(\lambda = 0)$.
- **Method:** Solve the conformal Einstein field equations on a conformally related manifold (M, g).
- **Observation:** If the conformal factor Θ is evolved along conformal geodesics in a vacuum spacetime then Θ is quadratic in the conformal time along the curves. Hence the location of the conformal boundary at $\Theta = 0$ can be prescribed in terms of initial data.
- **Part 1:** Proof of the stability of radiative spacetimes near i^+.
- **Part 2:** Extension of the results to the radiative Einstein-Maxwell case.

Setup

- The physical spacetime will be denoted by $(\tilde{M}, \tilde{g})$ and will be a 4-dimensional Lorentzian manifold $(+---)$ with sufficient smoothness.
- The vacuum equations with vanishing cosmological constant:

$$\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R} = 0 \qquad (1)$$

- The field equations are rewritten in terms of the conformally related "unphysical" metric

$$g_{\mu\nu} = \Theta^2 \tilde{g}_{\mu\nu}$$

and a g-orthonormal frame $\{e_k\}$.

Conformal geometry

A Weyl connection related to g and $\tilde{g}$ is a torsion-free connection $\hat{\nabla}$ such that

$$\begin{aligned} \hat{\Gamma}^{\rho}_{\mu\nu} &= \tilde{\Gamma}^{\rho}_{\mu\nu} + S_{\mu\nu}{}^{\rho\lambda} b_{\lambda} \\ &= \Gamma^{\rho}_{\mu\nu} + S_{\mu\nu}{}^{\rho\lambda} f_{\lambda} \end{aligned}$$

where $b_\mu = f_\mu + \partial_\lambda ln\,\Omega$ and

$$S_{\mu\nu}{}^{\rho\lambda} \equiv \delta_{\mu}{}^{\rho}\delta_{\nu}{}^{\lambda} + \delta_{\mu}{}^{\lambda}\delta_{\nu}{}^{\rho} - g_{\mu\nu}g^{\rho\lambda}.$$

We use the following shorthand notation $\quad \hat{\nabla} = \nabla + f = \tilde{\nabla} + b.$

The Schouten tensor of $\hat{\nabla}$ is defined as

$$\hat{P}_{\mu\nu} = \frac{1}{n-2}\left(\hat{R}_{(\mu\nu)} + \frac{n-2}{n}\hat{R}_{[\mu\nu]} - \frac{1}{2(n-1)}g_{\mu\nu}g^{\rho\lambda}\hat{R}_{\rho\lambda}\right).$$

Conformal geodesics

A conformal curve is a curve $x(\tau)$ with a triple (v^μ, b_ν, e_k^μ) satisfying

$$\tilde{\nabla}_v v^\rho + S_{\mu\nu}{}^{\rho\lambda} v^\mu v^\nu b_\lambda = 0,$$
$$\tilde{\nabla}_v b_\nu - \frac{1}{2} S_{\mu\nu}{}^{\rho\lambda} v^\mu b_\rho b_\lambda = \tilde{P}_{\mu\nu} v^\mu,$$
$$\tilde{\nabla}_v e_k^\rho + S_{\mu\nu}{}^{\rho\lambda} v^\mu e_k^\nu b_\lambda = 0,$$

where $v^\mu = \dot{x}^\mu$.
The Schouten tensor $\tilde{P}_{\mu\nu}$ transforms under $\check{\nabla} = \tilde{\nabla} + f$ as

$$\check{P}_{\mu\nu} = \tilde{P}_{\mu\nu} - \tilde{\nabla}_\mu f_\nu + \frac{1}{2} S_{\mu\nu}{}^{\rho\lambda} f_\rho f_\lambda.$$

For $\check{\nabla}$ the same conformal geodesics is described by the triple $(v, b - f, e_k)$.

Canonical conformal factor for vacuum

Along each curve of a smooth congruence define the conformal factor by $\frac{d\Theta}{d\tau} = \Theta b_\mu v^\mu$. Then

- Θ is given for $\tau \in I$ by
$$\Theta = \Theta_* + \dot{\Theta}_*(\tau - \tau_*) + \ddot{\Theta}_*(\tau - \tau_*)^2,$$
where a quantity with a subscript $*$ is constant along $x(\tau)$.
- $b_k(\tau) = b_\mu e_k^\mu = \Theta^{-1}\left(\dot{\Theta}, d_{a*}\right)$ for $\tau \in I$,
where $d_{a*} = \Theta b_a(\tau_0)$, for $a = 1,\ 2,\ 3$.
- For $g = \Theta^2 \tilde{g}$ have constant $g(e_i, e_k)$ along the curve.
- For $\hat{\nabla} = \tilde{\nabla} + b$, induced by the congruence,
$$\hat{\nabla}_v v^\mu = 0, \qquad \hat{P}_{\mu\nu} v^\mu = 0, \qquad \hat{\nabla}_v e_k^\mu = 0.$$

The location of the conformal boundary $\Theta = 0$ depends only on the initial data $\Theta_*,\ \dot{\Theta}_*,\ \ddot{\Theta}_*,\ \tau_*$.

Existence theorem [Friedrich]

- Construct a smooth congruence of conformal geodesics (v, b, e_k).
- Use the Weyl connection $\hat{\nabla}$ and the conformal factor induced by the congruence as a new gauge.
- The vacuum field equations are re-expressed in terms of the conformally rescaled metric g and split into propagation and constraint equations along the congruence. The constraints are propagated.
- Given initial data satisfying the conformal constraints one has local existence of a solution to the conformal field equations.
- This implies a solution to the vacuum field equations.

Initial data: static gives radiative

We use the following two results:

Multipole conjecture [Herbertson, Backdahl]

Given a series of multipole moments satisfying an appropriate convergence condition, there exists a static solution of the vacuum Einstein equations with precisely those multipole moments at i^0.

Friedrich

Given an analytic solution to the conformal static equations on a neighbourhood $\mathcal{B}_a(i) \subset \bar{\mathcal{S}}$ there exists a solution, $(\bar{h}_{\alpha\beta}, \bar{\Omega})$, to the time symmetric conformal vacuum constraints on $\mathcal{B}_a(i)$ satisfying:

(i) it is asymptotically Euclidean and regular in a neighbourhood $\mathcal{B}_a(i)$;

(ii) it has vanishing mass;

(iii) the spacetime curvature constructed from the constraints and $(\bar{h}_{\alpha\beta}, \bar{\Omega})$ is analytic in $\mathcal{B}_a(i)$

Constructing radiative vacuum spacetimes

- Specify a multipole sequence
- Obtain static initial data
- Transform into radiative initial data
- Specify suitable initial data for a congruence of conformal geodesics
- Construct a vacuum spacetime $(\mathring{\mathcal{M}}, \mathring{g})$.

We note

- The point i acts as timelike infinity for $(\mathring{\mathcal{M}}, \mathring{g})$.
- Local existence in τ for a neighbourhood of i corresponds to semi-global existence for the physical radiative vacuum spacetime.
- Vanishing multipoles give Minkowski.
- $(\mathring{\mathcal{M}}, \mathring{g})$ and the congruence serve as the reference spacetime for perturbations.

The conformal picture

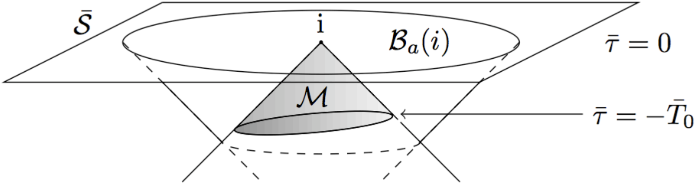

Figure: A conformal representation of the reference solution obtained as the development of Euclidean initial data on $\bar{\mathcal{S}}$. The shaded region represents the physical spacetime.

We have an infinite family of reference spacetimes, parametrised by the multipoles.

Perturbation

- Fix a slice $\mathcal{S} \equiv \{\tau = -\bar{T}_0\}$ in $(\mathring{\mathcal{M}}, \mathring{g})$ on which we get hyperboloidal data for our reference spacetime.
- Prescribe perturbed initial data for the spacetime on $\mathcal{S} \cap I^{-}(i^{+})$ satisfying the constraints and extending the data to $\mathcal{S}$.
- Give initial data for the congruence of conformal curves and $\Theta_*, \dot{\Theta}_*, \ddot{\Theta}_*$ such that the location of $\mathscr{I}$ is fixed to be the same as in the reference spacetime.
- Note: The congruence is tilted with respect to the hyperboloid. First evolve locally to be able to prescribe the full inital data for the perturbed spacetime.
- Use a stability theorem by Kato to prove that the evolution is stable up to $\mathscr{I}$.
- Show that $\mathscr{I}$ and i^{+} are regular.

Theorem

[LV 09] Suppose $m \geq 4$*. Let* $u_0 = \mathring{u}_0 + \breve{u}_0$ *be hyperboloidal initial data. Given* $T \in (\bar{\tau}_0, 2\bar{\tau}_0)$*, there exists* $\varepsilon > 0$ *such that:*

(i) *For* $||\breve{u}_0||_m < \varepsilon$ *there exist a unique solution* $u = \mathring{u} + \breve{u}$ *to the conformal propagation equations with minimal existence interval* $\tau \in [0, T]$ *and* $u \in C^{m-2}([0, T] \times S^3)$*.*

(ii) *The associated congruence of conformal geodesics contains no conjugate points in* $[0, T]$*.*

(iii) *Timelike infinity* i^+ *is located at* $(\tau_+, 0, 0, 0)$ *with* $\tau_+ = -\Omega/\dot{\Theta}_* \in [0, T]$ *.*

(iv) *The Hessian is nondegenerate at* i^+*.*

The solution $u = \mathring{u} + \breve{u}$ *on* $\mathcal{D}^+(\mathcal{S})$ *implies a* C^{m-2} *solution* $(\mathcal{M}, \tilde{g})$ *to the vacuum Einstein field equations with vanishing cosmological constant.*

The proof uses little details of the reference spacetime. Instead it relies properties related to the conformal geometry.

Setup 2

- For the Einstein-Maxwell case the field equations are

$$\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R} = -\lambda\tilde{g}_{\mu\nu} + \tilde{T}_{\mu\nu},$$
$$\tilde{T}_{\mu\nu} = \tilde{F}_{\mu\lambda}\tilde{F}^{\lambda}{}_{\nu} - \frac{1}{4}\tilde{g}_{\mu\nu}\tilde{F}_{\lambda\rho}\tilde{F}^{\lambda\rho},$$
$$\tilde{\nabla}^{\mu}\tilde{F}_{\mu\nu} = 0, \quad \tilde{\nabla}_{[\mu}\tilde{F}_{\nu\lambda]} = 0.$$

- Here we take $\lambda = 0$.
- Once more the field equations are written in terms of the conformally related "unphysical" metric $g_{\mu\nu} = \Theta^2\tilde{g}_{\mu\nu}$.

Simon [1992]

Given an analytic solution $(h_{\alpha\beta}, \Phi, \Psi)$ to the conformal electrostatic equations on a neighbourhood $\mathcal{B}_a(i) \subset \bar{S}$ there exists a solution, $(\bar{h}_{\alpha\beta}, \bar{\Omega}, \bar{E}_\alpha)$, to the time symmetric conformal Einstein-Maxwell constraints on $B_a(i)$ satisfying:

(i) it is asymptotically Euclidean and regular in a neighbourhood $\mathcal{B}_a(i)$;

(ii) it has vanishing mass and charge;

(iii) the spacetime curvature constructed from the constraints and $(\bar{h}_{\alpha\beta}, \bar{\Omega})$ is analytic in $\mathcal{B}_a(i)$

Plan: Construction of reference spacetime and proof of stability as for vacuum

Note we have no multipole result for EM.

Two approaches to the Einstein Maxwell case

A **Conformal geodesics**
Need to give evolution equations for Θ, b_k and some of their derivatives.

Price: Loss of control over location of conformal boundary. Require different proof for regularity.

B **A new class of conformal curves**
Use curves that recover the behaviour of Θ and b and the possibility to fix the location of the conformal boundary.

Price: New system of evolution equations and constraints.

A conformal curve will be a curve $x(\tau)$ with a triple (v^μ, b_ν, e_k^μ) satisfying

$$\tilde\nabla_v v^\rho + S_{\mu\nu}{}^{\rho\lambda} v^\mu v^\nu b_\lambda = 0,$$
$$\tilde\nabla_v b_\nu - \frac{1}{2} S_{\mu\nu}{}^{\rho\lambda} v^\mu b_\rho b_\lambda = \tilde H_{\mu\nu} v^\mu,$$
$$\tilde\nabla_v e_k^\rho + S_{\mu\nu}{}^{\rho\lambda} v^\mu e_k^\nu b_\lambda = 0,$$

where $v^\mu = \dot x^\mu$ and $\tilde H_{\mu\nu}$ is a tensor that transforms under $\check\nabla = \tilde\nabla + f$ as

$$\check H_{\mu\nu} = \tilde H_{\mu\nu} - \tilde\nabla_\mu f_\nu + \frac{1}{2} S_{\mu\nu}{}^{\rho\lambda} f_\rho f_\lambda.$$

For $\check\nabla$ the same conformal curve is described by the triple $(v, b - f, e_k)$. Independently of the connection we can define the tensor

$$\tilde J_{\mu\nu} \equiv \tilde P_{\mu\nu} - \tilde H_{\mu\nu} = \check P_{\mu\nu} - \check H_{\mu\nu} = P_{\mu\nu} - H_{\mu\nu}.$$

Canonical conformal factor

For trace-free matter set $\tilde{J}_{\mu\nu} = \frac{1}{2}\tilde{T}_{\mu\nu}$. Define the conformal factor by $\frac{d\Theta}{d\tau} = \Theta b_\mu v^\mu$ along each curve. Then

- $\tilde{H}_{\mu\nu} = \lambda \tilde{g}_{\mu\nu}$
- Θ and b_k take the same form as for conformal geodesics in vacuum

$$\Theta = \Theta_* + \dot{\Theta}_*(\tau - \tau_*) + \ddot{\Theta}_*(\tau - \tau_*)^2.$$

- In the connection $\hat{\nabla} = \tilde{\nabla} + b$ induced by the curves we get

$$\hat{\nabla}_v v^\mu = 0, \qquad \hat{P}_{\mu\nu} v^\mu = \tilde{J}_{\mu\nu} v^\mu, \qquad \hat{\nabla}_v e_k^\mu = 0.$$

- In the vacuum case ($\tilde{J}_{\mu\nu} = 0$) the curves reduce to conformal geodesics.

The setup to prove existence:

- Construct a smooth congruence of conformal curves (v, b, e_k).
- Use the Weyl connection $\hat{\nabla}$ and the conformal factor induced by the congruence as the new gauge.
- Express the Einstein-Maxwell field equations in terms of the conformally rescaled metric g. Split into propagation and constraint equations along the congruence. Show that the constraints are propagated.
- Write the overall system of PDEs in a symmetric hyperbolic form.
- Prove local existence of a solution to the conformal Einstein-Maxwell equations, given initial data satisfying the conformal constraints.
- This implies a solution to the physical Einstein-Maxwell equations.

Constructing radiative Einstein-Maxwell spacetimes

Simon [1992]

Given an analytic solution $(h_{\alpha\beta}, \Phi, \Psi)$ to the conformal electrostatic equations on a neighbourhood $\mathcal{B}_a(i) \subset \bar{\mathcal{S}}$ there exists a solution, $(\bar{h}_{\alpha\beta}, \bar{\Omega}, \bar{E}_\alpha)$, to the time symmetric conformal Einstein-Maxwell constraints on $B_a(i)$ satisfying:

(i) it is asymptotically Euclidean and regular in a neighbourhood $\mathcal{B}_a(i)$;

(ii) it has vanishing mass and charge;

(iii) the spacetime curvature constructed from the constraints and $(\bar{h}_{\alpha\beta}, \bar{\Omega})$ is analytic in $\mathcal{B}_a(i)$

- Given a solution $(\bar{h}_{\alpha\beta}, \bar{\Omega}, \bar{E}_\alpha)$, and suitable initial data for a congruence of conformal curves we can construct an Einstein-Maxwell spacetime $(\mathring{\mathcal{M}}, \mathring{g})$.
- We have local existence in τ for a neighbourhood of i. The causal past of i is conformal to a spacetime, for which i is timelike infinity. Hence we have semi-global existence for the physical radiative Einstein-Maxwell spacetime.
- $(\mathring{\mathcal{M}}, \mathring{g})$ serve as our reference spacetime for perturbations.

The semi-global stability theorem

Theorem

[LV 10] Let $u_0 = \mathring{u}_0 + \breve{u}_0$ *be hyperboloidal initial data for the vacuum conformal field equations on* $\mathcal{S}$*. Then*

- *Given* $\tau_+ \equiv -\Omega/\dot{\Theta}_*$ *and if* $\breve{u}_0$ *is sufficiently small, there exists on* $[0, \bar{\tau}_0] \times \mathcal{S}$ *a unique solution* $u = \mathring{u} + \breve{u}$ *to the conformal propagation equations such that the associated congruence contains no conjugate points in* $[0, \tau_+]$*.*
- *The solution* $u = \mathring{u} + \breve{u}$ *on* $\mathcal{D}^+(\mathcal{S})$ *implies a smooth solution* $(\mathcal{M}, \tilde{g})$ *to the Einstein field equations with vanishing cosmological constant, where* $\tilde{g}_{\mu\nu} = \Theta^{-2} g_{\mu\nu}$*.*
- *The spacetime* $(\mathcal{M}, \tilde{g})$ *has a conformal boundary given by the set of points for which* $\Theta = 0$*. The conformal boundary consists of the set* $\mathscr{I}$*, which represents future null infinity, and the point* $i^+ \equiv (\tau_+, 0, 0, 0)$*, which represents timelike infinity.*

Once more we use mostly properties related to the conformal geometry and very little detail of the reference spacetime.

Future work and open questions

- The hope is that a similar analysis can be used to investigate the stability of tracefree perfect fluids.
- As reference spacetime chose FRW. The setup of the congruence and the gauge as before. Deal with different kind of matter terms.
- Expectation that the conformal geometry will again serve as a good guide towards the proof. No or little detailed knowledge of the underlying physical spacetime.
- For isotropic singularity look for a quantity to indicate its location in the unphysical setting, analogous to conformal infinity being given by $\Theta = 0$.

Thank you for listening