Higher dimensional near-horizon geometries

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Motivation

- Higher dimensional black holes of interest in string theory and the gauge/gravity duality (AdS/CFT) for variety of reasons.
- Classification of $D > 4$ *stationary* black holes. Asymptotically flat, Kaluza-Klein and AdS (even $D = 4$) all of interest.
- Asymptotically flat especially important. Arises as a limit of other cases, e.g. black holes localised in KK dimensions.
- $D = 4$ uniqueness/no-hair theorem for Kerr-Newman black hole. Specified by (M, J, Q).

Motivation

- $D > 4$: much richer space of stationary black hole solutions to Einstein's equations. Black hole *non-uniqueness*.
- Asymptotically flat vacuum black holes: Myers-Perry S^{D-2} (analogue of Kerr), black rings $S^1 \times S^2$ [Emparan, Reall '01]
- Key issues: horizon topology? rotational symmetries? Number of solutions?
- *Extremal* black holes provide a simplified setting to address some of these questions. May give clues to general problem.

Motivation – extremal black holes

- Extremal black holes special in quantum gravity – zero Hawking temperature.
- Tend to admit simple statistical derivation of Bekenstein-Hawking entropy (e.g. within String Theory).
- Extremal case often excluded from general theorems. E.g. uniqueness theorem for extremal Kerr only recently shown.
- Extremal black holes have a well defined notion of a "near-horizon geometry".

Higher dimensional black holes

- Uniqueness of asymptotically flat *static* black holes. As in 4D, Schwarzschild-Tangherlini only vacuum soln [Gibbons et al '02].

- Interesting asymptotically flat solutions must be non-static. Typically this means they rotate.

- Conserved charges: mass M, angular momenta J_i where $i = 1, \ldots [(D-1)/2]$, and charges from any Maxwell fields.

- Black hole non-uniqueness: fixing these conserved charges insufficient to fix black hole solution.

Higher dimensional black holes

- *Weyl solutions*: vacuum black holes with $R \times U(1)^{D-3}$ symmetry [Emparan, Reall '01]. Integrable system as in 4D.
- Only for $D = 4, 5$ symmetry compatible with asymptotic flatness. ($D > 5$ with KK asymptotics)
- Much progress for $D = 5$ asymptotically flat vacuum black holes with $R \times U(1)^2$ symmetry:
  - Uniqueness theorem: there is at most one non-extremal black hole, with a connected horizon, for given M, J_i and *rod data*. [Hollands, Yazadjiev '07]
  - Multi black holes solns constructed: black saturn etc. [Elvang, Figueras '07]

Horizon topology

- Spatial cross-section of event horizon is a compact manifold H of dimension $D-2$. Assume connected.
- Dominant energy $\implies$ Yamabe(H) > 0 [Galloway & Schoen '05]. $D=5: H = S^3, S^1 \times S^2$ ($+$ quotients & connected sums)
- Asymptotically flat (and globally AdS): H must be (oriented)-cobordant to S^{D-2}. Non-trivial for $D > 5$.
- $D > 5$ only example known $H = S^{D-2}$ (Myers-Perry). What other topologies are actually realised?

Rigidity theorem

- Killing horizon is a null hypersurface with a normal V which is a Killing vector field. $V^2 = 0$ on horizon.

- Event horizon of a (non-degenerate) *stationary* black hole is a Killing horizon. *Rotating* $\implies$ $R \times U(1)^s$ sym with $s \geq 1$.
[Hawking '72; Hollands, Ishibashi, Wald '06; Isenberg, Moncrief '83 '08]

- Asymptotically flat $\implies$ $s \leq \text{rank } SO(D-1) = [(D-1)/2]$. Known solns saturate upper bound – e.g. $D = 5$ with $s = 2$

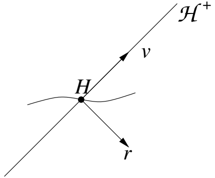
- We will assume rigidity theorem in degenerate case. $D = 4$ it has been shown, partial results for $D > 4$. [Hollands, Ishibashi '08]

Degenerate horizons

- Surface gravity κ defined by $d(V^2) = -2\kappa V$ on horizon. Degenerate (or extremal) horizon is such that $\kappa = 0$.

- Near Killing horizon use Gaussian null coords: $V = \frac{\partial}{\partial v}$, horizon $r = 0$, x^a coords on *compact* cross-section H.

$$g = r\, f(r,x) dv^2 + 2 dvdr + 2r\, h_a(r,x) dvdx^a + \gamma_{ab}(r,x) dx^a dx^b$$



- $\kappa = 0$ equivalent to $f(r,x) = r\, F(r,x)$.

Near-horizon geometry

- Metric near an extremal horizon in Gaussian null coordinates:

$$g = r^2 F(r,x) dv^2 + 2 dv dr + 2r\, h_a(r,x) dv dx^a + \gamma_{ab}(r,x) dx^a dx^b$$

- Near-horizon limit [Reall '02]: $v \to v/\epsilon$, $r \to \epsilon r$ and $\epsilon \to 0$. Limit is *near-horizon geometry* (NHG).

$$g_{NH} = r^2 F(x) dv^2 + 2 dv dr + 2r\, h_a(x) dv dx^a + \gamma_{ab}(x) dx^a dx^b$$

- Near-horizon data (γ_{ab}, h_a, F) all defined on H, r-dependence fixed.

- New symmetry: $v \to v/\lambda$, $r \to \lambda r$. Together with $v \to v + c$ these form 2d non-abelian group.

General comments

- NHG of extremal black hole *solution* to some theory of gravity, must also be a solution. Classify these!

- Gives potential horizon *topologies* and *geometries* of full extremal black hole solns. Can rule out black hole topologies.

- Information outside black hole horizon lost. Existence of NHG soln does not guarantee existence of corresponding black hole.

- Einstein eqs for NHG equivalent to Einstein-like eqs on H. Problem of compact Riemannian geometry in $D-2$ dims.

Examples

- Einstein-Maxwell: extremal Reissner-Nordstrom $M = Q$, near-horizon limit is $AdS_2 \times S^2$ (homogeneous, static)

$$ds^2 = Q^2[-r^2dv^2 + 2dvdr + d\Omega_2^2]$$

- Vacuum: extremal Kerr $J = M^2$, NH limit is S^2-bundle over AdS_2. Isometry $SO(2,1) \times U(1)$ (inhomogeneous, non-static)

$$\begin{aligned} ds^2 &= \frac{(1+\cos^2\theta)}{2}\left[-\frac{r^2dv^2}{2a^2} + 2dvdr + a^2d\theta^2\right] \\ &+ \frac{2a^2\sin^2\theta}{1+\cos^2\theta}\left(d\phi + \frac{rdv}{2a^2}\right)^2 \end{aligned}$$

- $D > 4$ many *extremal* examples... all have $SO(2,1)$ isometry!

Near-horizon symmetries

[Kunduri, JL, Reall '07]

- Is $SO(2,1)$ symmetry generic? Not obvious, in general one only has 2d symmetry in (v,r)-plane.
- Static case can show NHG is a (warped) product of AdS_2 (or $R^{1,1}, dS_2$) and H, i.e. $g_{NH} = \Gamma(x)\eta_2 + \gamma(x)$
- In general need Einstein eqs. + symmetry assumptions. Assume $U(1)^{D-3}$ rotational symmetry:

$$\gamma = d\rho^2 + \gamma_{ij}(\rho)d\phi^i d\phi^j \qquad h = \Gamma^{-1}(k_i(\rho)d\phi^i - \Gamma'(\rho)d\rho)$$

where $m_i = \partial/\partial\phi^i$ generate sym.

Near-horizon symmetries

[Kunduri, JL, Reall '07]

- $D = 4, 5$ Einstein-Maxwell-scalar theory with non-positive scalar potential (+ higher derivative terms).

- **Theorem**: NHG of extremal black hole *solution* with $R \times U(1)^{D-3}$ symmetry, has $SO(2,1) \times U(1)^{D-3}$ symmetry.

- **Proof**: integrating certain components of Einstein eqs, and redefining $r \to \Gamma(\rho) r$ gives

$$g_{NH} = \Gamma(\rho)[-r^2 dv^2 + 2dvdr] + d\rho^2 \\ +\gamma_{ij}(\rho)(d\phi^i + k^i rdv)(d\phi^j + k^j rdv)$$

AdS_2-isometries $rdv \to rdv + df$ act on NHG $\phi^i \to \phi^i - k^i f$.

Near-horizon symmetries

- NHG is a fibration of H over AdS_2. Generic orbits are T^{D-3}-fibrations over AdS_2 (cohomogeneity-1).

- $U(1)^{D-3}$-invariant Maxwell fields also invariant under $SO(2,1)$.

- $D > 5$ known examples outside validity of theorem. NHG of extremal Myers-Perry has $SO(2,1) \times U(1)^{[(D-1)/2]}$ symmetry.

- $D > 5$. Can prove theorem for cohom-1 non-abelian rotational symmetry G, such that $U(1)^{[(D-1)/2]} \subset G$ [Figueras et al '08]

Vacuum near-horizon geometries

- Focus on vacuum Einstein eqs $R_{\mu\nu} = \Lambda g_{\mu\nu}$ with $\Lambda \leq 0$. For NHG *equivalent* to solving eqs on H:

$$Ric(\gamma)_{ab} = \frac{1}{2} h_a h_b - \nabla_{(a} h_{b)} + \Lambda \gamma_{ab}$$

- This determines γ_{ab}, h_a. Remaining NH data F then given by $F = \frac{1}{2} h^a h_a - \nabla_a h^a + \Lambda$.

- Difficult to solve. Has only been solved under extra symmetry assumptions.

- Static NHG: $h = d\lambda$. Compactness of $H \implies h \equiv 0$. [Chrusciel, Reall, Tod '05]. Interesting vacuum solns must be non-static.

$D = 4$ near-horizon geometries

- Assuming $U(1)$ rotational symmetry renders horizon eqs ODEs (cohomogeneity-1 geometry).

- General *axisymmetric* vacuum NHG solution is near-horizon limit of extremal Kerr/Kerr-AdS
  [Hajicek '73; Lewandowski, Pawlowski '03; Kunduri, JL '08]

- Result generalises to Einstein-Maxwell-Λ [Kunduri, JL '08]. NHG extremal Kerr-Newman-AdS in only axisymmetric soln.

  Note: static NHG can be classified with no assumptions: $AdS_2 \times S^2$ (or T^2, H^2 in $\Lambda < 0$ case)

$D = 5$ near-horizon geometries

[Kunduri, JL '08]

- Classification of vacuum $U(1)^2$-invariant NHG.
- $U(1)^2$ symmetry restricts topology of H. Must be $S^1 \times S^2$, S^3, Lens spaces (or T^3 which cannot arise from black holes).
- NHG eqs reduce to ODEs on 1d orbit space $H/U(1)^2 \cong$ interval. Key step: coord x defined by $dx = i_{m_1} i_{m_2} \epsilon_3$.
- Result: all such vacuum NHG solns arise from *known* extremal black holes (either asympt flat or KK)!
    - S^3: Myers-Perry; 2 types of KK black holes [Rasheed '95]
    - $S^1 \times S^2$: boosted Kerr string; black ring [Pomeransky, Senkov '06]

$D = 5$ near-horizon geometries

- $R_{\mu\nu} = \Lambda g_{\mu\nu}$. Classification can be reduced to 6th order non-linear ODE. Only S^3 solution known (Myers-Perry-AdS).
- Adding Maxwell field complicates classification. No electro-magnetic duality; local dipole charges...
- Even static NHG not fully classified. Electric case: warped product of AdS_2 and inhomogeneous S^3 (incl. $AdS_2 \times S^3$)! [Kunduri, JL '09]
- Supersymmetric case understood in general for $\Lambda = 0$ [Reall '02] and with $U(1)^2$ for $\Lambda < 0$ [Kunduri, JL, Reall '06].

Uniqueness of vacuum black holes with rotational symmetry

- Uniqueness theorems for vacuum black holes with $R \times U(1)^{D-3}$ symmetry can be proved (Weyl solutions).

- 2d orbit space $B = \mathcal{M}/(R \times U(1)^{D-3})$. $\partial B = \{$ horizon, axes, infinity $\}$. Rod data is extra data defined on ∂B.

- Einstein eqs reduce to elliptic PDE on B. Boundary-value problem, with BC determined by M, J_i and rod data.

- Extremal black holes usually not considered in such uniqueness theorems. Extremality enters BC near horizon boundary.

Uniqueness of extremal vacuum black holes

- $SO(2,1) \times U(1)^{D-3}$-sym NHG in Weyl coordinates (ρ, z): (connected) horizon $r = 0$ appears as a point [Figueras, JL '09]

$$\rho = r \sin\theta \qquad z = r\cos\theta$$

where $x = \cos\theta$ parameterises $H/U(1)^{D-3}$ = interval.

- Can extend existing vacuum black hole uniqueness theorems to extremal case using good coords (r, x).

- Uniqueness of extremal Kerr. Fills a gap in no-hair theorem! [Meinel et al '08; Amsel et al '09; Figueras, JL '09; Chrusciel, Nguyen '10]

- 5D: uniqueness of asympt flat, $U(1)^2$ extremal vacuum black holes, given J_i and rod structure. [Figueras, JL '09]

$D > 5$ near-horizon geometries

- Can determine all vacuum $U(1)^{D-3}$–NHG [Hollands, Ishibashi '09]. Tops: $S^3 \times T^{D-5}$, $S^2 \times T^{D-4}$. T^{D-2} ruled out [Holland '10].
- Asympt flat black holes must have rotational symmetry contained in $SO(D-1)$. Look for NHG with such sym.
- Number of commuting rotational KVF $\leq [(D-1)/2] < D-3$. Not in Weyl class for $D > 5$. Classification not yet possible.
- Myers-Perry give examples of vacuum NHG with $H = S^{D-2}$ and $U(1)^{[(D-1)/2]}$ rotational symmetry.

$D > 5$ near-horizon geometries

- Myers-Perry strings gives examples of vacuum NHG with $H = S^1 \times S^{D-3}$. D odd has $U(1)^{[(D-1)/2]}$ rotational sym!

  Conjecture: NHG of *tensionless* string corresponds to yet to be found asymptotically flat black ring [Figueras et al '08]

- Einstein-Maxwell: $AdS_2 \times H$, H any positive Einstein space. Not expected to arise from asymptotically flat black hole!

- Focus on non-static and vacuum near-horizon geometries. Can we find new examples with appropriate symmetry?

New infinite class of $D > 5$ near-horizon geometries

[Kunduri, JL'10]

- $D = 2n + 2$: have found new NHG solutions to $R_{\mu\nu} = \Lambda g_{\mu\nu}$ with $\leq n = [(D-1)/2]$ commuting rotational KVF.
- H is inhomogeneous S^2-bundle over any compact positive Kähler-Einstein base manifold K.
- For fixed base, specified by one continuous param L (spin) and an integer $m > p > 0$ ($p =$ Fano index of K).
- All H cobordant to S^{2n} and positive Yamabe type. Candidates for NHG of new black holes!

New infinite class of $D > 5$ near-horizon geometries

[Kunduri, JL'10]

- "Calabi Ansatz": (g, J) is Kahler-Einstein structure on base K, where $2J = d\sigma$, $Ric(g) = 2ng$ and

$$\gamma = d\rho^2 + B(\rho)^2(d\phi + \sigma)^2 + A(\rho)^2 g$$
$$h = C(\rho)(d\phi + \sigma) + \lambda'(\rho)d\rho$$

  New solns are of this form (not necessarily most general)

- NHG of this form always have $SO(2,1) \times U(1) \times G$ symmetry, where G is symmetry of KE base. [Figueras et al '08]

- Local form of solns, with $K = CP^{n-1}$, include NHG of Myers-Perry $a_i = a$. $H = S^{2n}$ with $SU(n) \times U(1)$ sym.

$D = 6$ near-horizon geometries

- $D = 6$: H is S^2 bundle over $\mathbb{CP}^1 \cong S^2$. Topology classified by $\pi_1(SO(3)) = Z_2$. One non-trivial bundle.

- $m > 2$: m even $S^2 \times S^2$; m odd $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ (i.e. 1-pt blow-up of $\mathbb{CP}^2$). Metrics cohomogeneity-1 with $SU(2) \times U(1)$ sym.

- Remark: analogous to Page metric, which is an Einstein metric on $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ with $m = 1$.

- Note: simply connected closed 4-manifolds with $U(1)^2$ action must be connected sums of $\mathbb{CP}^2$, $\overline{\mathbb{CP}^2}$, $S^2 \times S^2$, S^4.
  [Orlik, Raymond '70]

$D > 6$ near-horizon geometries

- $D > 6$: different m gives different topology. Infinite number of topologies for fixed KE base.

- Many choices for KE base, e.g. $KE = \mathbb{CP}^{n-1}$. If KE base is toric, the horizon has $U(1)^n$ rotational symmetry.

- If KE base has no (continuous) isometries get NHG with exactly $U(1)$ rotational symmetry!

  E.g. $KE = \mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$ for $4 \leq k \leq 8$. Further $k \geq 5$ have moduli space: extra continuous parameters.

- If there are corresponding black holes must have $R \times U(1)$ symmetry. Saturate lower bound of rigidity theorem!

Open problems

- Complete classification of $R \times U(1)^2$ vacuum black holes in 5D, i.e. what rod data leads to regular black holes?
- Classification of 5D $U(1)^2$-invariant NHG with Maxwell fields. Reduces to complicated algebraic problem. [Kunduri, JL]
- 5D BH/NHG with exactly $U(1)$ rotational symmetry. Other applications: KK black holes, brane-world black holes
- Uniqueness theorems for AdS black holes? AdS black rings? Find NHG with black ring topology and $\Lambda < 0$.
- $D > 5$. Black holes with non-spherical horizons? Classification of NHG with appropriate symmetries?

Summary

- Near-horizon geometries can be used to learn about geometry and topology of horizons of extremal black holes.
- Much progress in 4D/5D: classification of NHG, uniqueness theorems for extremal black holes.
- $D > 5$ black holes poorly understood. Examples of possible black hole NHG with new horizon topology.
- Many interesting problems remain in higher dimensions...