Properties of Black Di-ring

IGUCHI, Hideo (Nihon University)

based on the collaboration with Prof Mishima
(arXiv:1008.4290)

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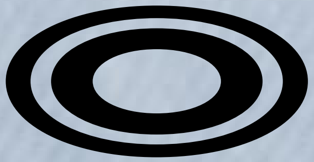
Outline

- Solution generations of black di-ring
- Relatoion between black di-ring I and II
- Phase structure of black di-ring
- Black di-ring in thermodynamic equilibrium
- Thermodynamic stability (instability)
- Summary

Solution generations of black di-rings

- Black di-ring is concentric configuration composed of two S^1-rotating black ring



- The solution was discovered by using solitonic method similar to Bäcklund transformation (HI and T Mishima, PRD 75, 064018, (2007)) (black di-ring I)
- The di-ring solution was reconstructed by using the inverse scattering method (Evslin and Krishnan, CQG 26, 125018, (2009)) (black di-rinh II)

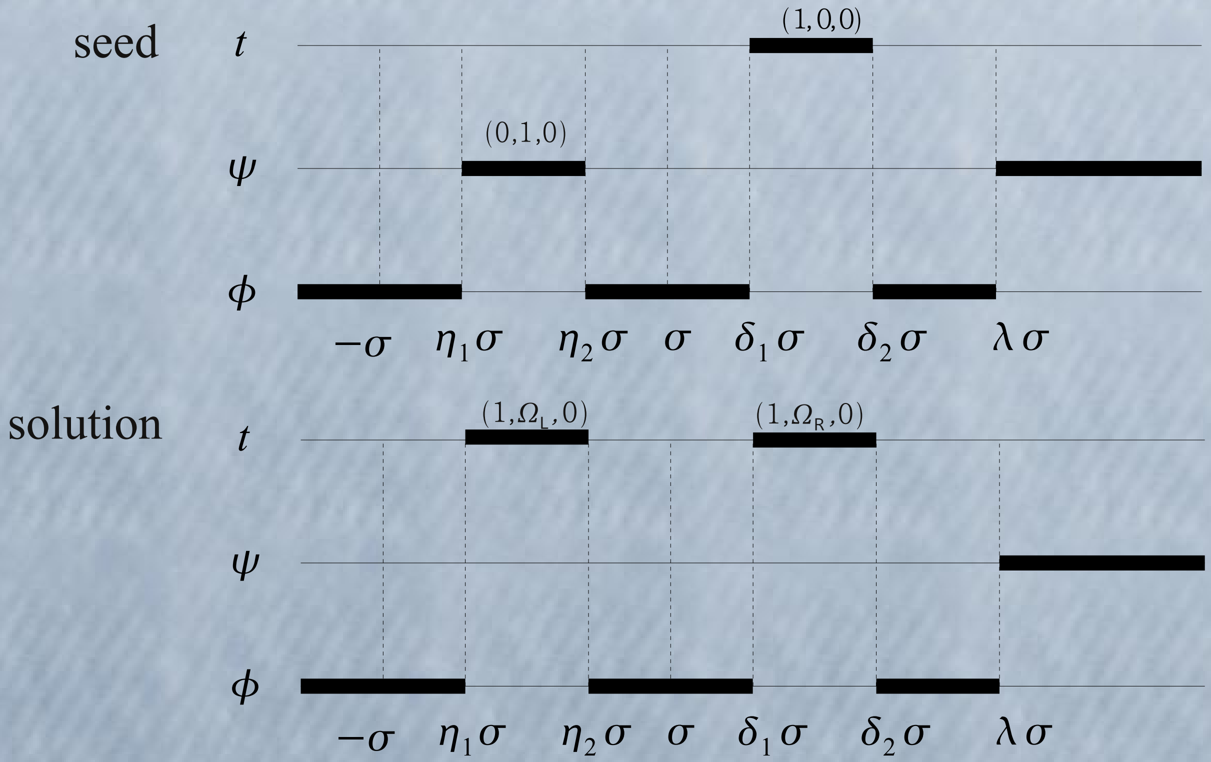
- The black di-ring solutions were constructed by rather different solitonic solution-generation technique, so that the expressions of the solutions are considerably different.

- Therefore it would be a difficult task to determine whether the two di-ring systems are equivalent or not with straightforward algebraic calculation.

- In the following we will explain the equivalence of these two di-ring solutions by the help of the work by Hollands and Yazadjiev for the uniqueness of higher dimensional black hole.

- In the analysis, it would be helpful to construct the two solutions by the same solitonic method. We generate the two representations of di-ring solution by the inverse scattering method and then show that there is an onto and one-to-one mapping between them.

Black di-ring I (original)



1. Finite spacelike rod with (0,1,0)

   ↓

   Finite timelike rod with $(1,\Omega_{\rm L},0)$

2. Finite timelike rod with (1,0,0)

   ↓

   Finite timelike rod with $(1,\Omega_{\rm R},0)$

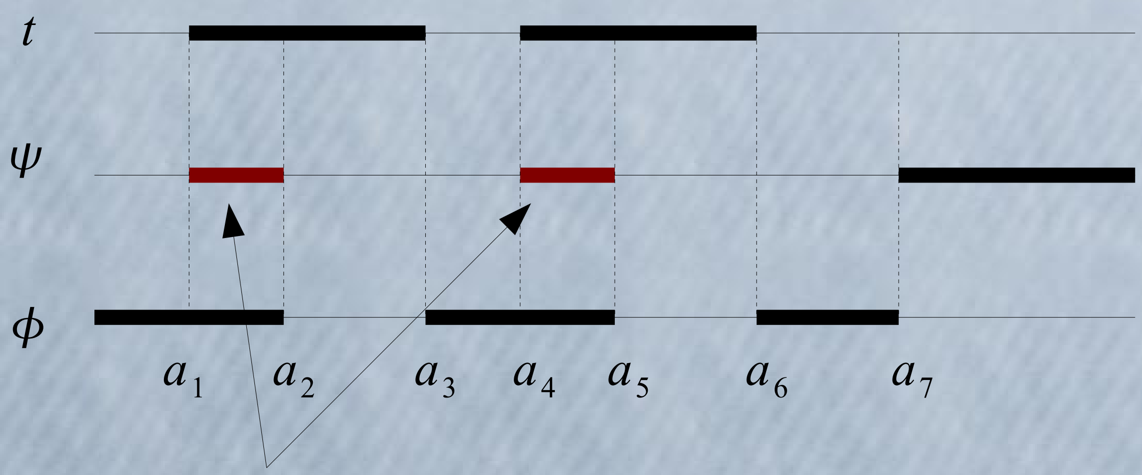
In the transformation, (anti-)solitons are added at $z = -\sigma$ and σ.

The non-trivial components of direction vectors are given by

$$\Omega_{\rm L}=\frac{(1+\alpha\beta)(2\beta(\delta_1+1)(1+\eta_2)-\alpha(\lambda+1)(\delta_2+1)(1-\eta_1))}{2\sigma^{1/2}(2\alpha\beta(\delta_1+1)(1+\eta_2)-(\lambda+1)(\delta_2+1)(\alpha^2(1-\eta_1)+2\alpha\beta+2))} \qquad \Omega_{\rm R}=-\frac{2\beta(1+\alpha\beta)}{\sigma^{1/2}((\lambda-1)\alpha\beta+\lambda+1)((\delta_2-1)\alpha\beta+\delta_2+1)}$$

where $\alpha=\pm\sqrt{\dfrac{2(\delta_1-1)(1-\eta_2)}{(\lambda-1)(\delta_2-1)(1-\eta_1)}}$, $\beta=\pm\sqrt{\dfrac{(\lambda+1)(\delta_2+1)(1+\eta_1)}{2(\delta_1+1)(1+\eta_2)}}$

Black di-ring II



negative line density

Procedure of modified ISM
(Pomeransky, 2006)

i.) remove solitons with trivial BZ-parameters
ii.) scale appropriately the metric
iii.) construct a generating matrix and recover the solitons with nontrivial BZ-parameters
iv.) scale back the metric

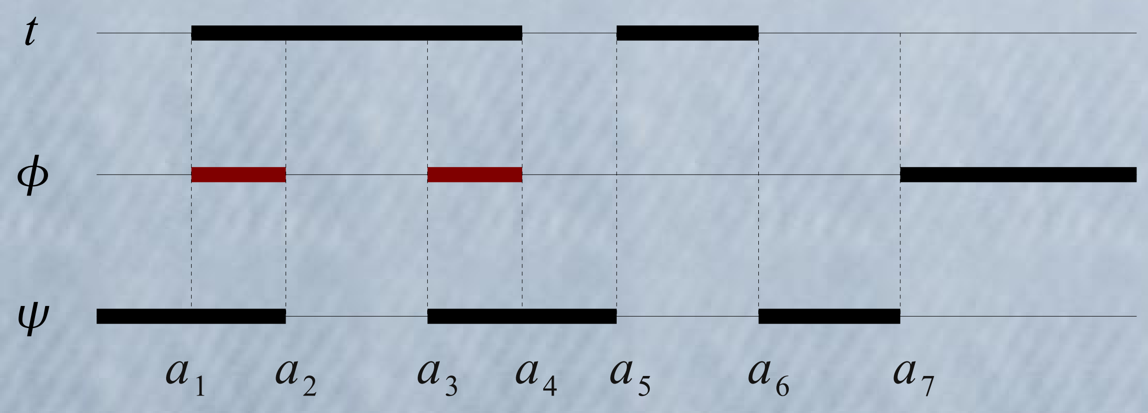
We remove and add (anti-)solitons at $z = a_1$ and a_4.

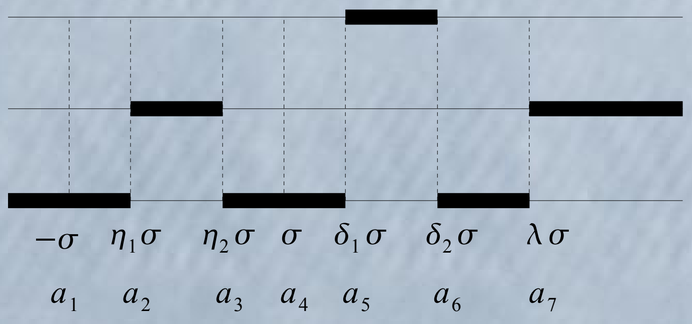
The BZ-parameters are fixed to remove singular behaviors on the axis.

For the solution generated by Evslin and Krishnan, the BZ-parameters should be given by

$$b_{\mathrm{L}}^{(\mathrm{II})} = \pm\left(\frac{2\,a_{31}\,a_{61}\,a_{71}}{a_{21}\,a_{51}}\right)^{1/2}, \qquad b_{\mathrm{R}}^{(\mathrm{II})} = \pm\left(\frac{2\,a_{43}\,a_{64}\,a_{74}}{a_{42}\,a_{54}}\right)^{1/2}$$

Black di-ring I (ISM)





The equivalence of original and ISM di-ring I is easily understood by the following fact.

After the step ii) the metric essentially coincide with the seed metric of original black di-ring I. And then, the (anti-)solitons are added at $z=a_1$ and a_4 in the step iii).

The relations between the parameters are trivial, for example, $a_1=-\sigma$, $a_2=\eta_1\sigma$, and so on.

Even if a_i's of di-ring I and II are equivalent each other, the solutions are different in general because the positions of negative density rods are different.

Equivalence of di-ring I and II

For the single horizon case, a solution is uniquely determined when the interval structure and ADM angular momentum are fixed. (Holland and Yazadjiev, 2008)

It seems that the proof of the above theorem can be applied to a multi-horizon system with some modifications. It would be insufficient if we fix the ADM angular momentum. We need fix the Komar angular momenta.

For the black di-ring we are considering, two different system become isometric when all the lengths of finite rods and Komar angular momenta coincide with each other.

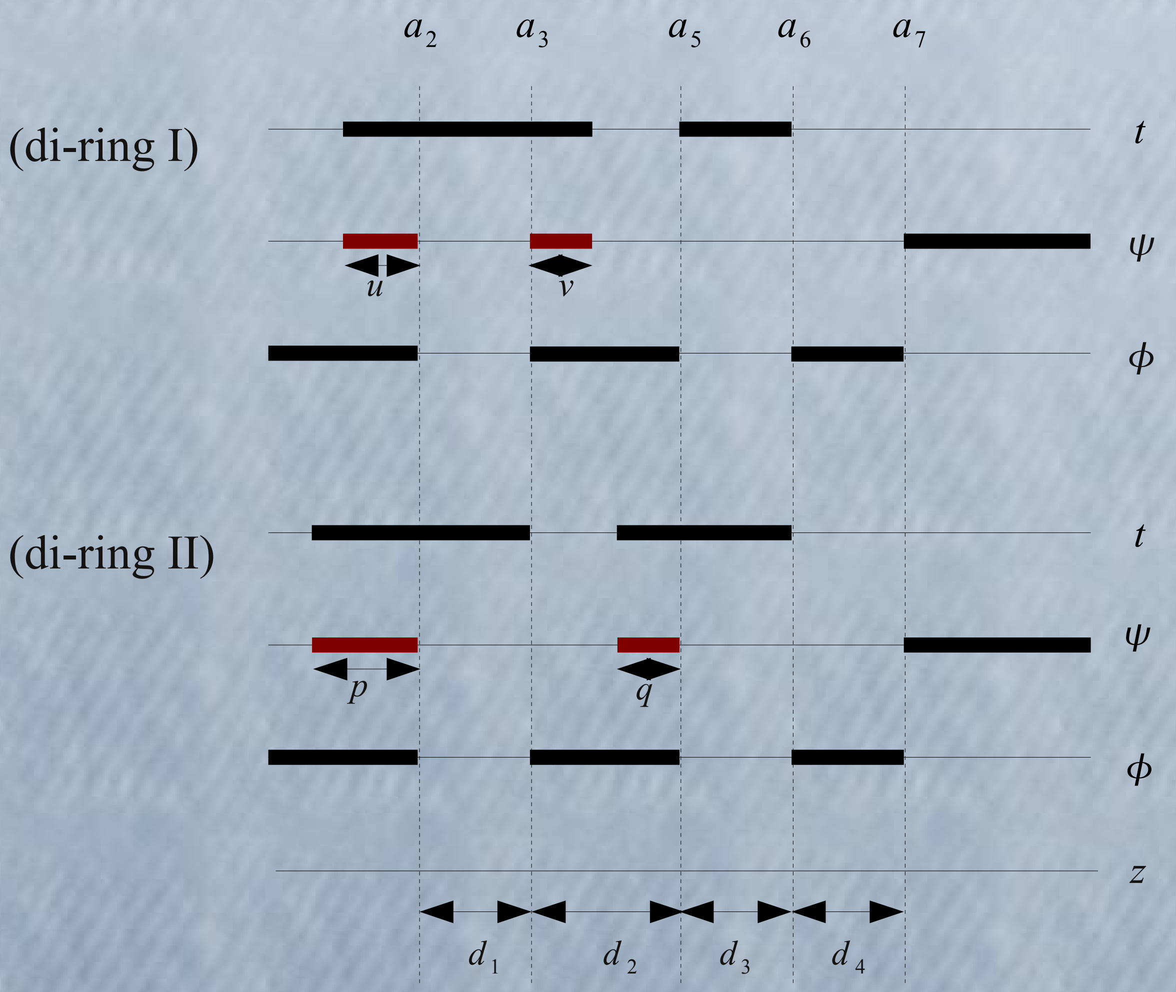
We can replace the Komar angular momenta with other two independent physical quantities. In the analysis we equate the ADM masses and the periodic angles of inner spaces of black di-ring I and II.

In conclusion, two di-rings are isometrically equivalent if the following conditions hold,

$$(d_1, d_2, d_3, d_4)^{(\mathrm{I})} = (d_1, d_2, d_3, d_4)^{(\mathrm{II})}, \qquad M^{(\mathrm{I})} = M^{(\mathrm{II})}, \qquad \Delta\phi_{\mathrm{R}}^{(\mathrm{I})} = \Delta\phi_{\mathrm{R}}^{(\mathrm{II})}$$

with the appropriate choice of signs of BZ parameters.

Mapping between black di-ring I and II



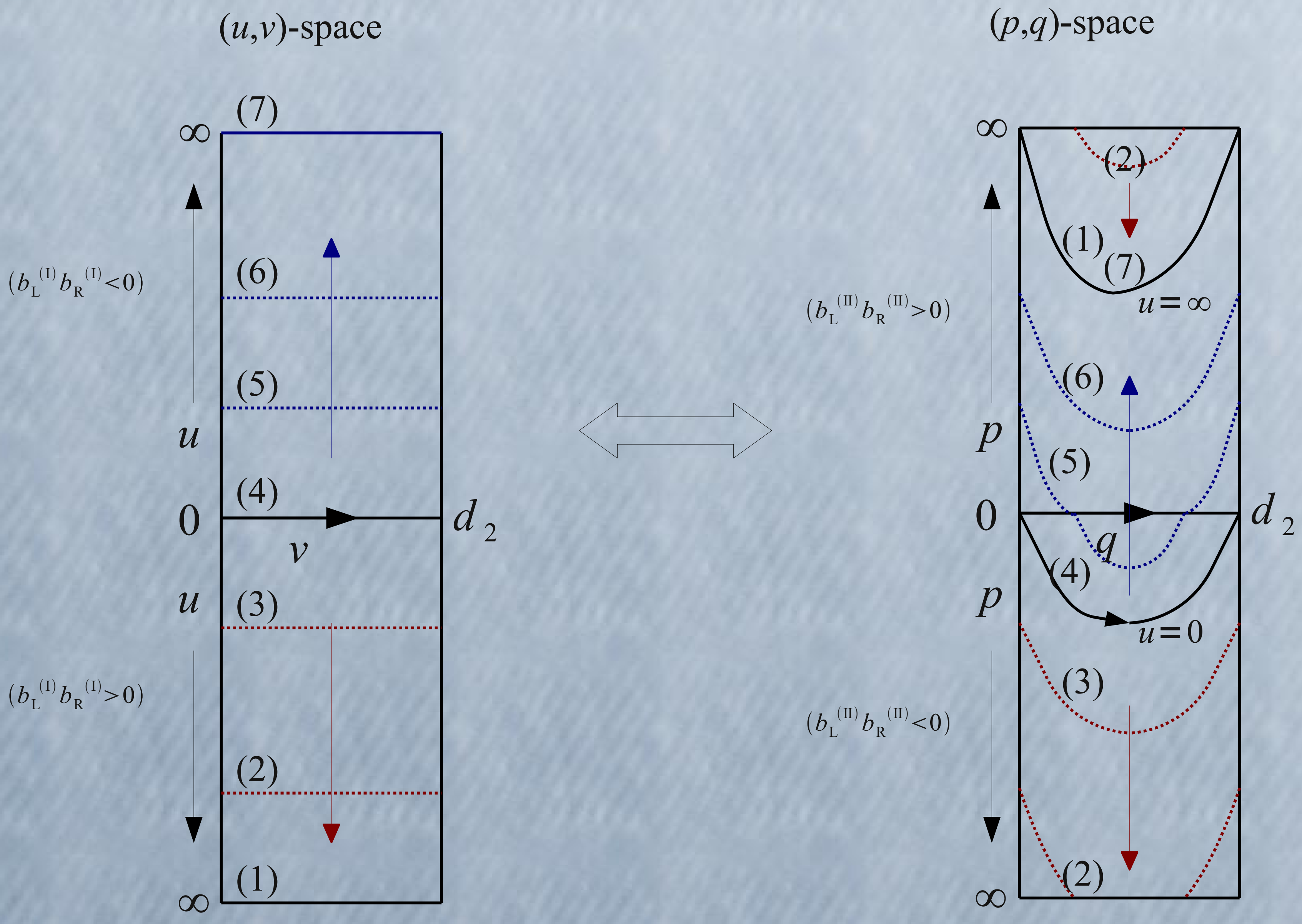
The mathematical relation ship between black di-ring I and II are given by solving the simultaneous equations,

$$M^{(\mathrm{I})}(u,v) = M^{(\mathrm{II})}(p,q)$$
$$\Delta\phi_{\mathrm{R}}^{(\mathrm{I})}(u,v) = \Delta\phi_{\mathrm{R}}^{(\mathrm{II})}(p,q)$$

As a result, we can provide a mapping between (u,v) and (p,q).

Examining the mapping function, we can confirm that the correspondence between (u,v)-space and (p,q)-space is onto and one-to-one.

Schematic picture of correspondence between (u,v) and (p,q)



Phase structure of black di-ring

Here we investigate the region of the phase diagram covered by regular black di-ring.

The physical variables are normalized by ADM mass

$$j^2=\frac{27\pi}{32G}\frac{J^2}{M^3},\quad a_{hi}=\frac{3}{16}\sqrt{\frac{3}{\pi}}\frac{A_{hi}}{(GM)^{3/2}},\quad \tau_i=\sqrt{\frac{32\pi}{3}}T_i(GM)^{1/2},\quad \omega_i=\sqrt{\frac{8}{3\pi}}\Omega_i(GM)^{1/2}$$

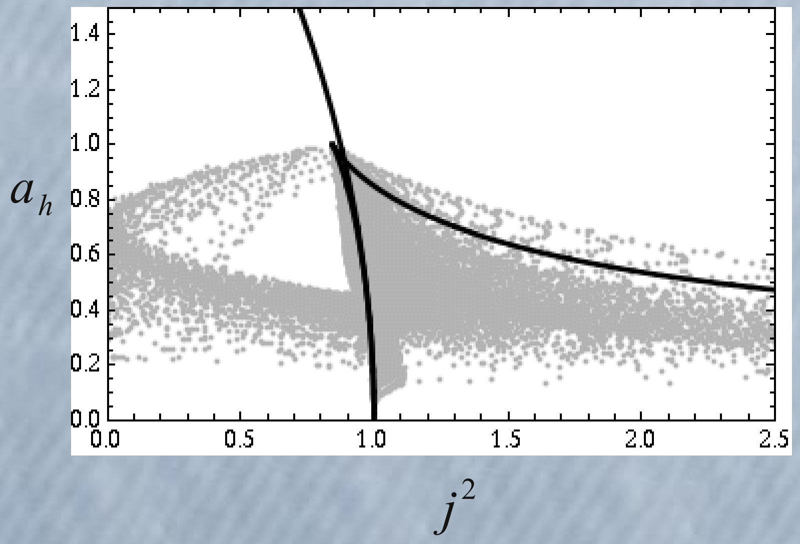
We use the representation of di-ring II.

There are 3 independent parameters for balanced di-ring because of one scaling freedom and 2 balance conditions.

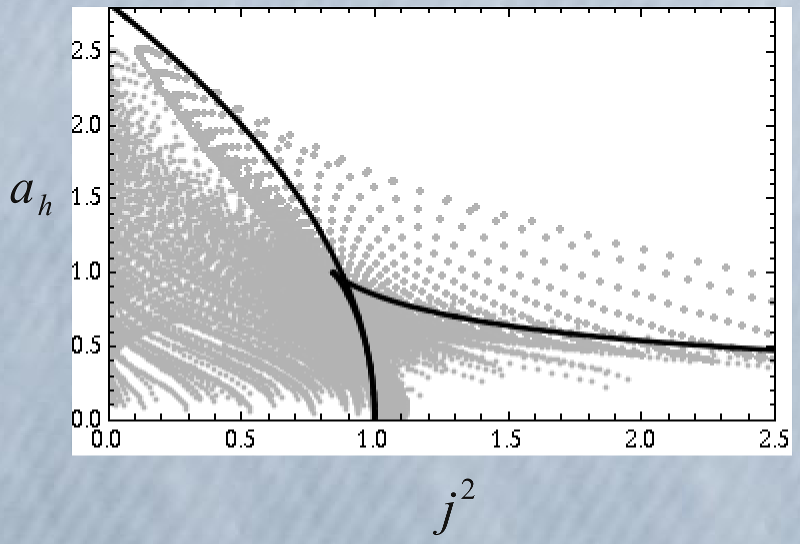
We choose d_2, d_3 and d_4 as these 3 parameters.
d_1 is determined by fixing the scaling as $d_1 + d_2 + d_3 + d_4 = 1$
p and q are determined by the balance conditions $\Delta\phi_L=2\pi$ and $\Delta\phi_R=2\pi$

Plot of (j^2, a_h) corresponding to the sets of (d_2, d_3, d_4) chosen by random systematic sampling

Black di-ring



Black Saturn



- The upper bound of total area of BD is same as the black ring a_h=1.
- There are BD configurations with total area greater than BR with the same j^2.
- The solution with j=0 is possible as similar as BS.
- The low entropy BD is scarcely distributed except j^2=1.
(There may be a lower bound of total area.)

Black di-ring in TDE

·Here we consider the black di-ring in thermodynamic equilibrium (thermodynamic black di-ring) in which the horizons have equal temperature and angular velocity.

·Such configuration is possible for black Saturn. (Elvang, et. al., 2007)
·It was argued that a multi-ring in thermodynamic equilibrium is difficult because a single black ring is uniquely determined by fixing its temperature and angular velocity.

·We show the existence of black di-ring in thermodynamic equilibrium and examine its properties.

·Similar results have been independently obtained by Emparan and Figueras (arXiv:1008.3243).

The thermodynamic black di-ring should satisfy the following four conditions,

$$\tau_{\mathrm{L}}=\tau_{\mathrm{R}}, \quad \omega_{\mathrm{L}}=\omega_{\mathrm{R}}, \quad \left(\frac{\Delta\phi_{\mathrm{L}}}{2\pi}\right)^2=1, \quad \left(\frac{\Delta\phi_{\mathrm{R}}}{2\pi}\right)^2=1\,.$$

These 4 equations can be reduced to the following nearly solved equations,

$$0=(4h_1^2-7h_1h_4+4h_4^2)h_3^3-(8h_1^3-8h_1^2h_4-h_1h_4^2++4h_4^3)h_3^2+(5h_1^4-2h_1^3h_4-2h_1^2h_4^2+h_1h_4^3+h_4^4)h_3-h_1^5$$

$$h_2=h_1-h_3+h_4$$

$$u-v=\frac{(h_1-h_3)(h_2^2+2h_1h_3-h_3^2)}{h_3h_4}$$

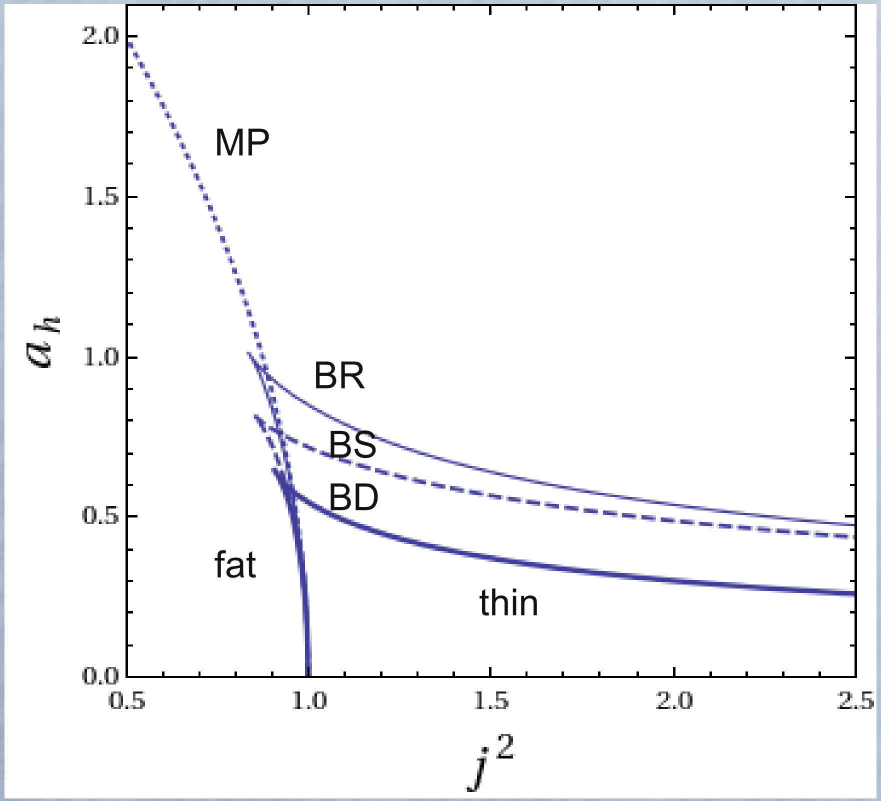
$$(u+h_1)(h_2-v)=\frac{h_1^2h_2^2}{h_3h_4}$$

where $h_1=d_1+d_2+d_3+d_{4,}\quad h_2=d_2+d_3+d_{4,}\quad h_3=d_3+d_{4,}\quad h_4=d_4$

By using global scaling, we fix $h_1=1$. We choose h_4 as a free parameter.
Then the other parameters are determined by using the above four equations.

The 2rd equation means that the lengths of horizon rods are equal with each other $d_1=d_3$

a_h v.s. j^2

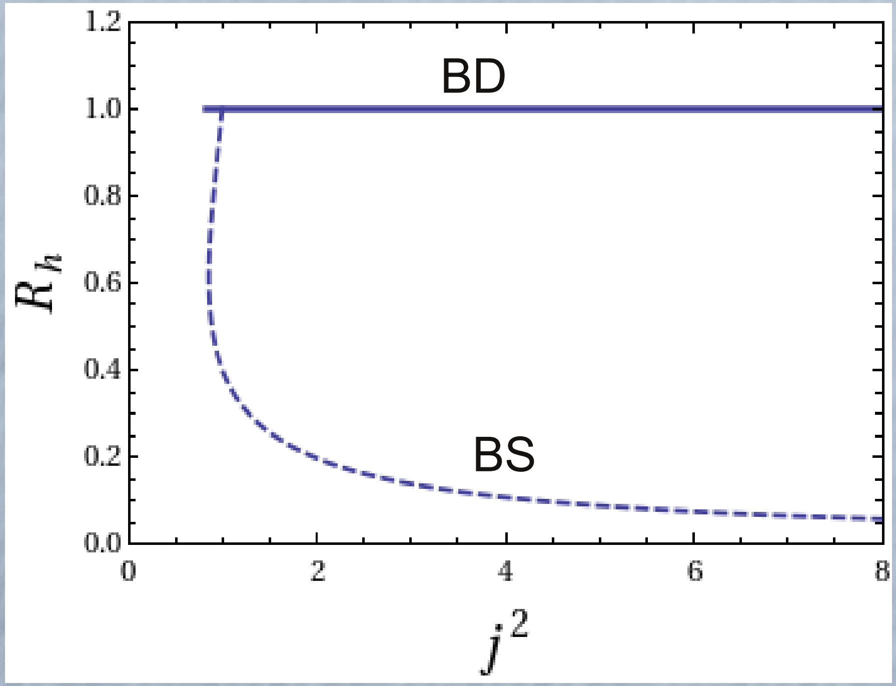


- BD (thick curve) appears as a continuous curve with cusp ($j_2 \approx 0.92075$ and $a_h \approx 0.62254$)

- The cusp divides the curve into fat and thin branches
- BD appears below BR (thin) and BS (dashed).
- In the large angular momentum limit, the are of BD is a half of BR

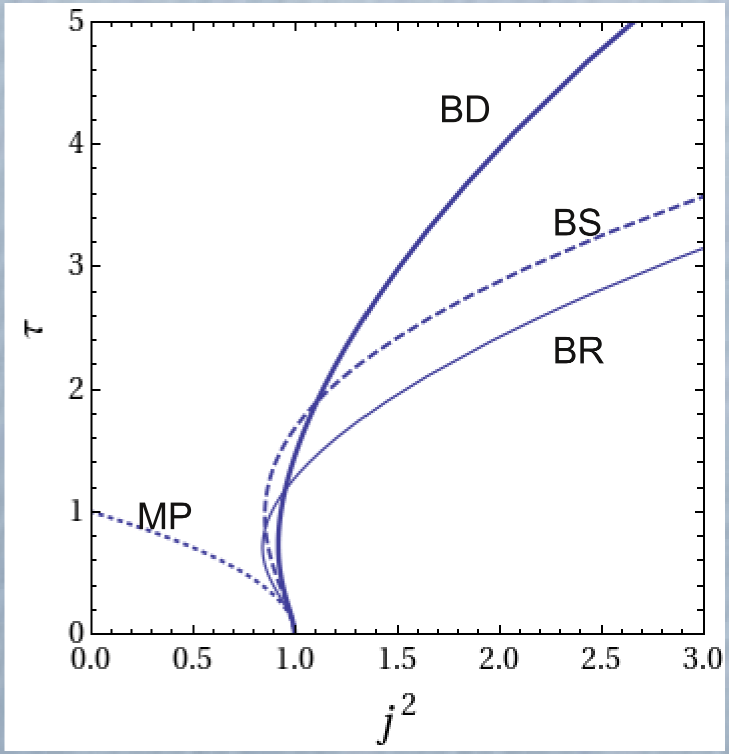
$$a_{h\mathrm{BD}} \approx \frac{1}{2} a_{h\mathrm{BR}}$$

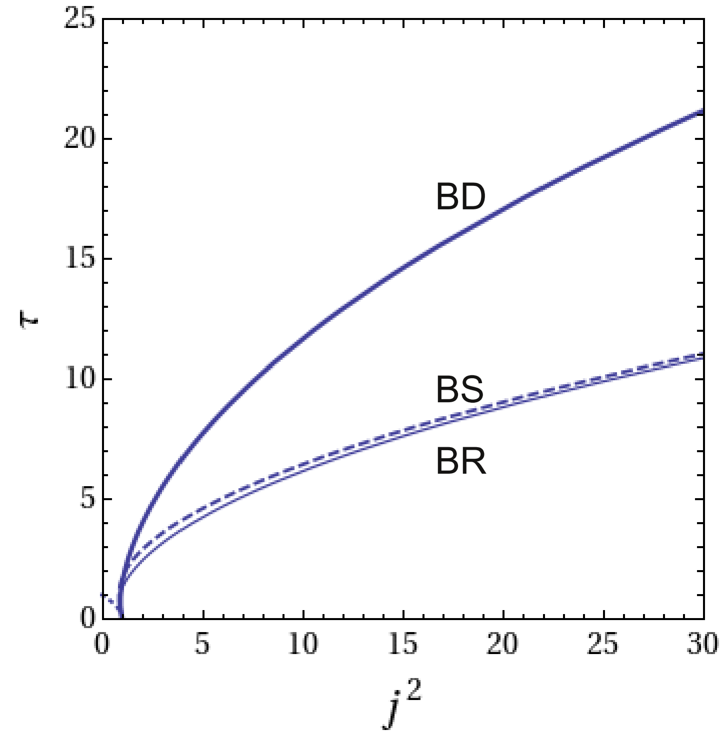
Ratio of area of inner object to outer ring $R_h = \frac{a_{h\mathrm{R}}}{a_{h\mathrm{L}}}, \quad \frac{a_{h\mathrm{hole}}}{a_{h\mathrm{ring}}}$



- The inner and outer ring of the black di-ring have the same area of horizons
- In the large j^2 limit, most of entropy of BS is in the ring.

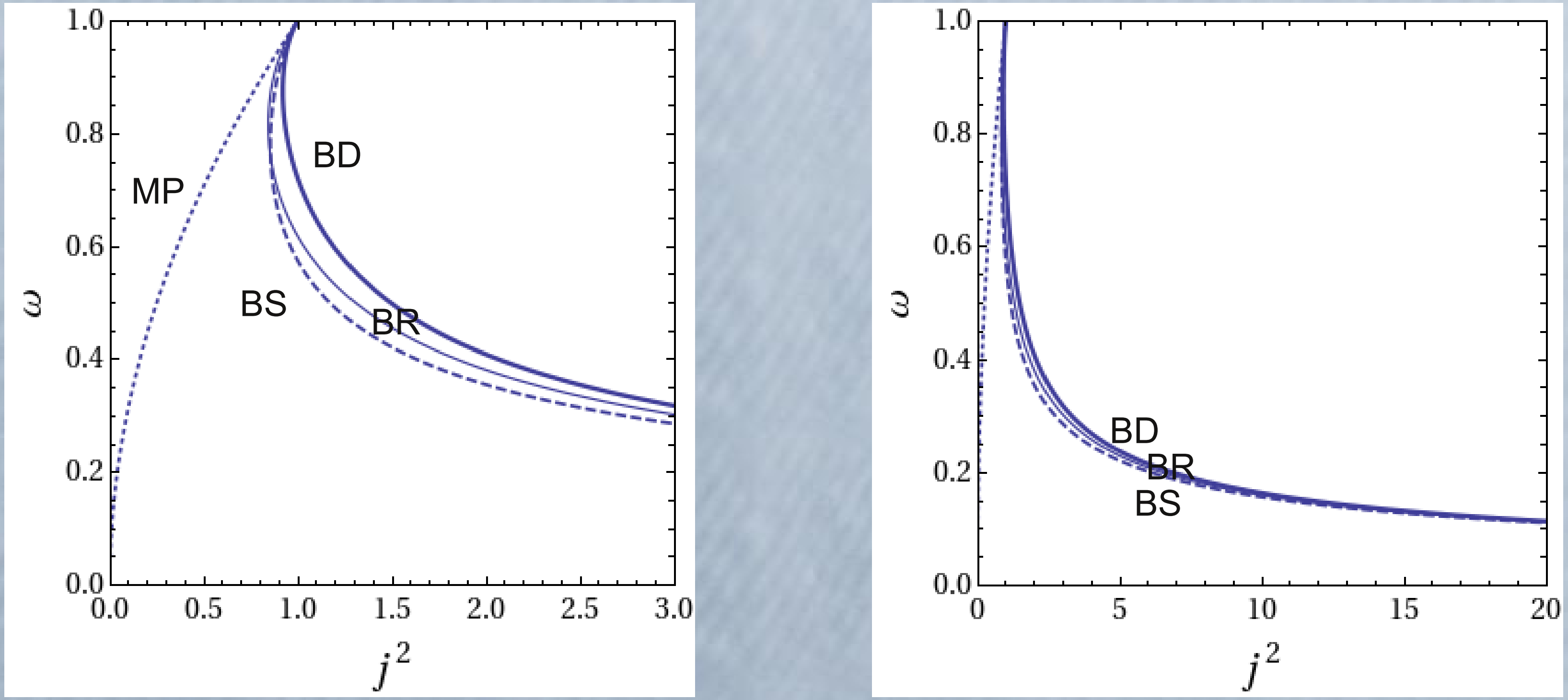
τ v.s. j^2





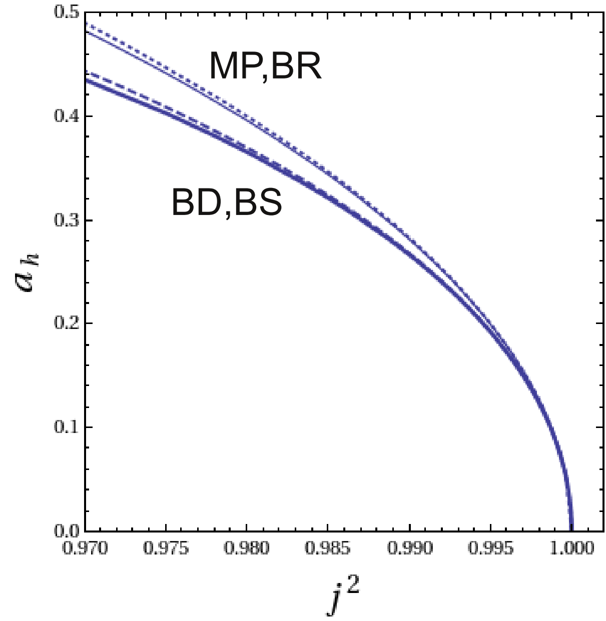
As j^2 increases, the temperature of BD approaches to the twice of BR (BS).

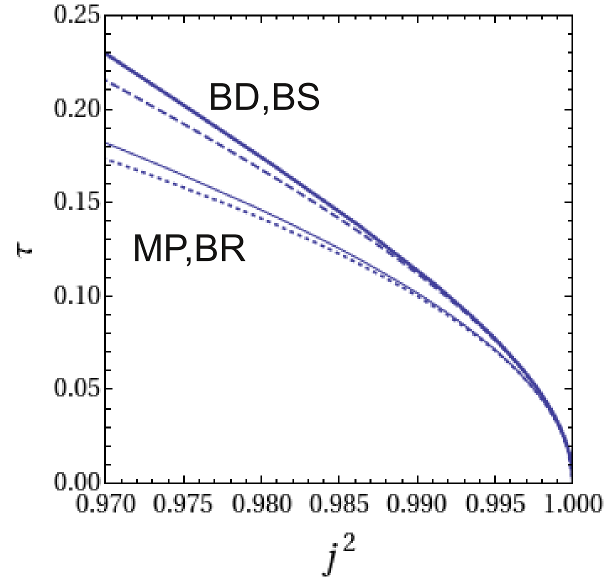
ω v.s. j^2

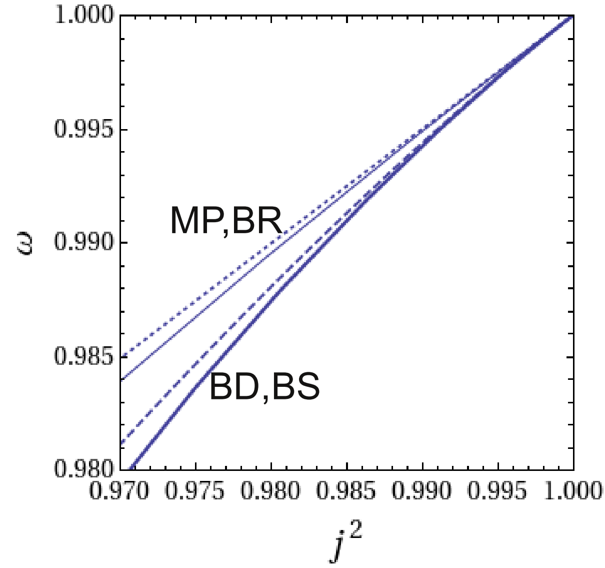


As j^2 increases, the angular velocity of BD approaches to the BR (BS).

Behaviors around $j^2 = 1$







The four curve separate into two pairs (BD, BS) and (MP, BR) around extremal MP limit.

Thermodynamic instability

- The black di-ring in TDE is not thermodynamically stable because the entropy is not the maximum.
- There is a possibility that the TDE is local maximum of entropy.
- It was found that the metastability of BS occurs when $0.85483<j^2<0.85494$ of thin ring branch. (Evsiln and Krishnan 2009)
- Here we investigate whether TDE of BD is metastable or not.

- The problem is constrained extremization of area a_h under the condition that j^2 is fixed.
- There are two additional constraints of balance conditions $\Delta\phi_{\mathrm{L}}=2\pi$ and $\Delta\phi_{\mathrm{R}}=2\pi$
- The system has six parameters and we can eliminate two of them by scaling freedom and one of the balance conditions.

The problem is extreme value problem of the function

$$f(x_1,x_2)=a_h(x_1,x_2,\varphi_3(x_1,x_2),\varphi_4(x_1,x_2))$$

φ_3 and φ_4 are obtained by simultaneously solving the constraints for x_3 and x_4.

$$g_1(x_1,x_2,x_3,x_4)=j^2(x_1,x_2,x_3,x_4)-j_0^2=0$$

$$g_2(x_1,x_2,x_3,x_4)=\Delta\phi_{\mathrm{R}}(x_1,x_2,x_3,x_4)-2\pi=0$$

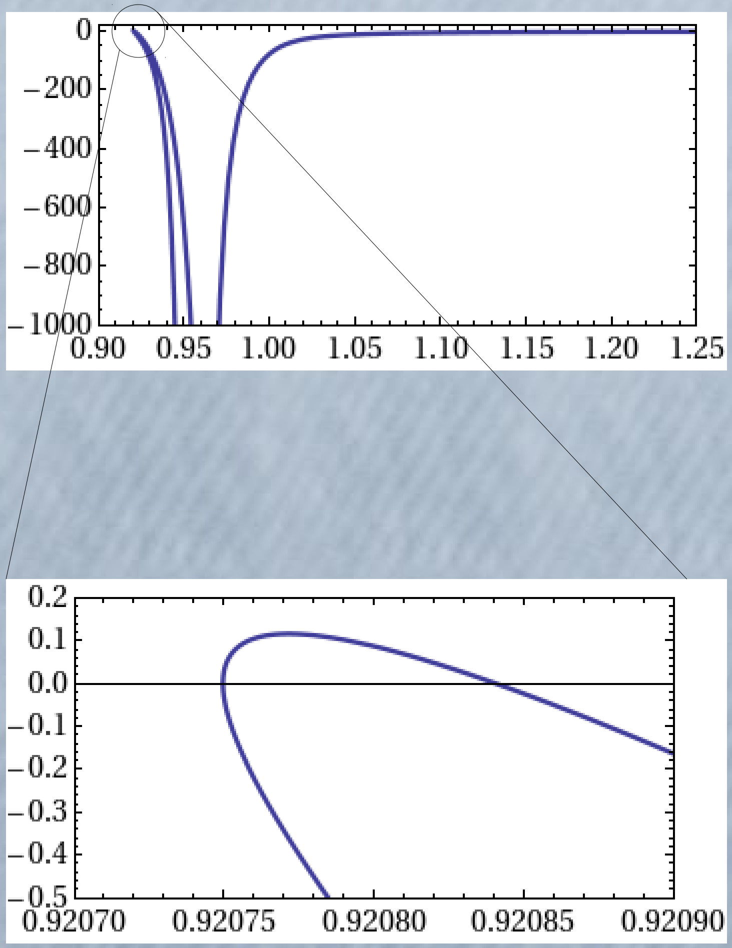
We numerically confirmed that the area of BD in TDE is extremal where the partial derivatives of $f(x_1, x_2)$ w.r.t x_1 and x_2 vanish.

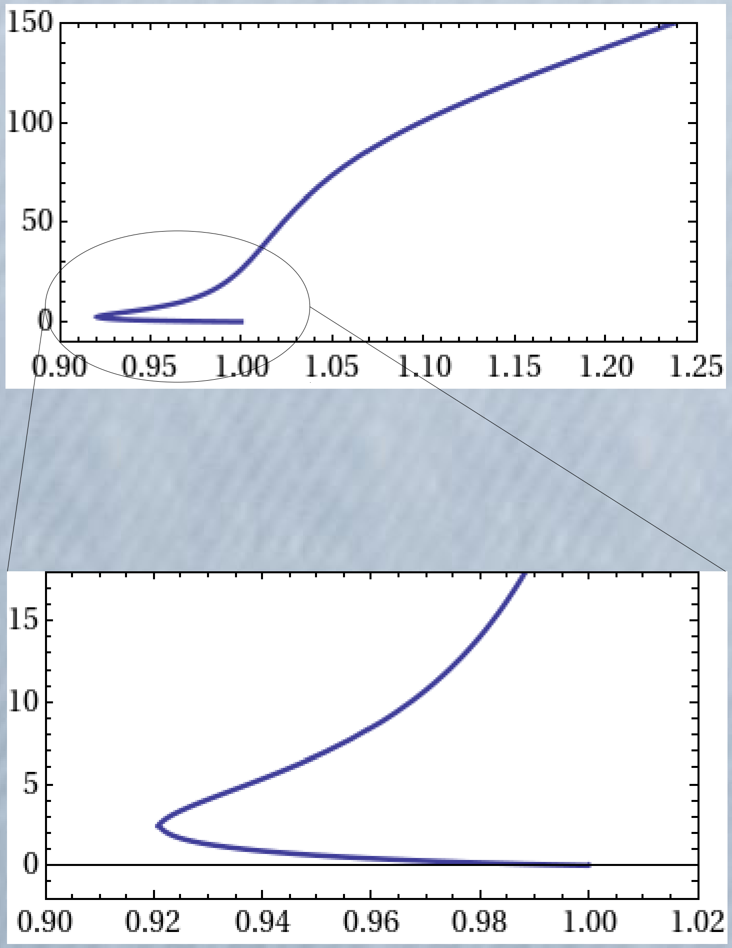
The TDE is metastable when the matrix

$$H_{ij}\equiv\frac{\partial^2 f(x_1,x_2)}{\partial x_i\partial x_j}$$

has only negative eigenvalues.

Eigenvalues vs j^2





One of the eigenvalues is always positive. There is no window of the metastability.

There is a narrow window $0.92075 < j^2 < 0.92084$ of thin ring branch where the both eigenvalues are positive.

Summary

- We have shown the equivalence between black di-ring I and II obtained by different solution generation methods.
- In the phase diagram, there appears a black di-ring in thermodynamic equilibrium.
- The thermodynamic equilibrium of black di-ring is not local maximum of the entropy and so is not metastable.

Bäcklund transformation

5D metric with 1 timelike and 2 spacelike Killing vectors

$$ds^2 = -e^{S-T}(dt - \omega\, d\phi)^2 + e^{-(S+T)}\rho^2\, d\phi^2 + e^{2\gamma-(S+T)}(d\rho^2 + dz^2) + e^{2T}\, d\psi^2$$

Einstein's equations

$$\left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \partial_z^2\right)T = 0$$

$$\left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \partial_z^2\right)S = -\frac{1}{\rho^2}e^{2S}((\partial_\rho\omega)^2 + (\partial_z\omega)^2)$$

$$\partial_\rho(\rho^{-1}e^{2S}\partial_\rho\omega) + \partial(\rho^{-1}e^{2S}\partial_z\omega) = 0$$

$$\partial_\rho\gamma = \frac{3}{4}\rho((\partial_\rho T)^2 - (\partial_z T)^2) + \frac{1}{4}\rho((\partial_\rho S)^2 - (\partial_z S)^2) - \frac{1}{4}\rho^{-1}e^{2S}((\partial_\rho\omega)^2 - (\partial_z\omega)^2)$$

$$\partial_z\gamma = \frac{3}{2}\rho\,\partial_\rho T\partial_z T + \frac{1}{2}\rho\,\partial_\rho S\partial_z S - \frac{1}{2}\rho^{-1}e^{2S}\partial_\rho\omega\,\partial_z\omega$$

From the Einstein equations we can obtain the equation for the Ernst potential

$$(\varepsilon_S + \varepsilon_S^*)(\partial_\rho^2\varepsilon_S + \rho^{-1}\partial_\rho\varepsilon_S + \partial_z^2\varepsilon_S) = 2(\partial_\rho\varepsilon_S\partial_\rho\varepsilon_S + \partial_z\varepsilon_S\partial_z\varepsilon_S)$$

where $\varepsilon_S \equiv e^S + i\Phi$ $\quad (\partial_\rho\Phi, \partial_z\Phi) = \rho^{-1}e^{2S}(-\partial_z\omega, \partial_\rho\omega)$

To solve the nonlinear Ernst equation, we apply the mathematical techniques developed in the 4 dimensional gravity, solitonic Neugebauer or Hoenselaers-Kinnersley-Xantopoulos transformation.

Rod structure analysis

R Emparan and H S Reall, Phys. Rev. D **65** (2002) 084025
S Harmark, Phys. Rev. D **70** (2004) 1240002

D-dimensional metric

$$ds^2 = G_{ab}\,dx^a\,dx^b + e^{2\nu}(d\rho^2 + dz^2)$$

Matrix field *G* satisfies $G^{-1}\nabla^2 G = (G^{-1}\nabla G)^2$ with the constraint $\det G = -\rho^2$

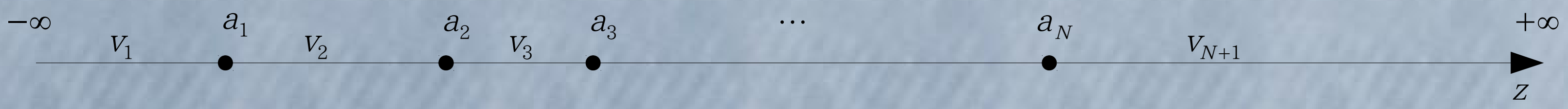
G(*0,z*) has at least one zero eigenvalue.

Regularity requires that only one eigenvalue is zero except at the isolated points

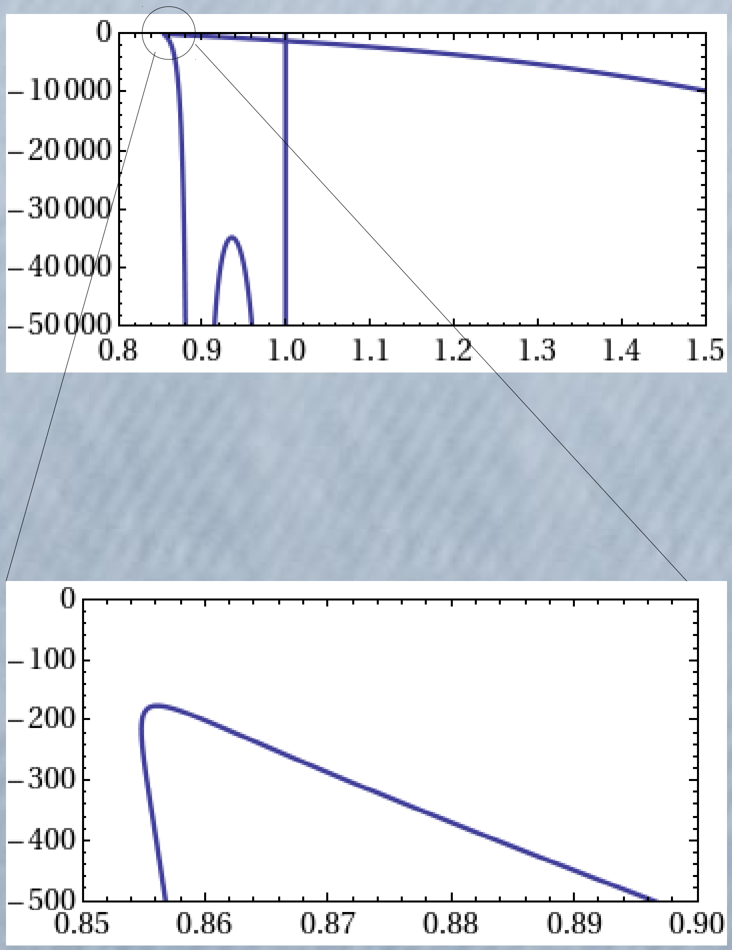
These isolated points divide the axis into intervals. Each interval is called a rod.

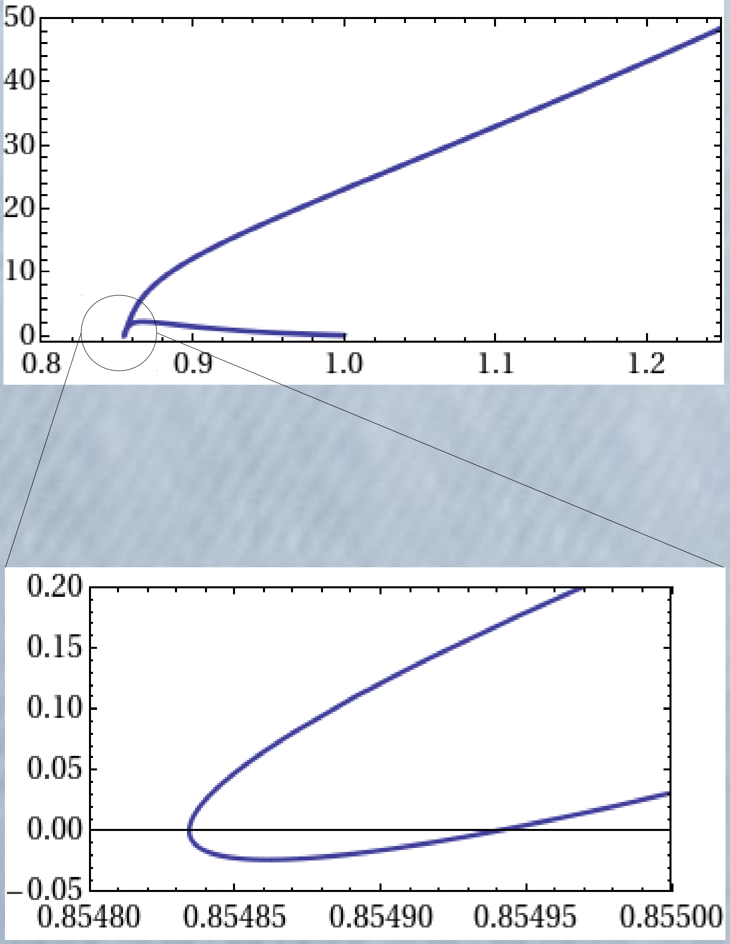
Each rod has direction vector *v* such that $G(0,z)\,v = 0$ for $z \in [a_i, a_{i+1}]$

The rod is called timelike (spacelike) if $\lim_{\rho\to 0} \frac{G_{ab}v^a v^b}{\rho^2}$ is negative (positive).



Eigenvalues vs j^2 (black Saturn)





One of the eigenvalues is always negative.

There is a narrow window $0.85483 < j^2 < 0.85494$ of thin ring branch where the both eigenvalues are negative.