Asymptotic Behavior of Spacetimes Approaching a Schwarzschild solution

Gustav Holzegel

Department of Mathematics

Princeton University

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The black hole stability problem

Formulated in the context of the Cauchy-problem (vacuum) [Choquet-Bruhat, Choquet-Bruhat and Geroch]

$$(\Sigma, h, K) + \text{constraints} \to (\mathcal{M}, g) \quad \text{satisfying } R_{\mu\nu} = 0 \tag{1}$$

Rough formulation of the stability question: Do sufficiently small perturbations of initial data for a Kerr solution lead to

1. spacetimes with similar global properties (regular event horizon, complete null-infinity) [= orbital stability]
2. spacetimes approaching a (different) Kerr solution for late times [=asymptotic stability]

Of course (2) implies (1). The point is that, in general, one does not expect to prove (1) without proving (2) at the same time.

An example of the above concepts: stability of Minkowski space [Christodoulou, Klainerman, Lindblad-Rodnianski, Bieri]. A key step was to understand linear field equations (spin 0, spin 1, spin 2) on Minkowski space and their decay.

This suggests to study the decay mechanisms of linear waves on fixed black hole backgrounds as a key step to the stability problem of black holes. This is an active area of research. Key to most approaches is the vectorfield method. For $\Box_g \psi = 0$:

$$T_{\mu\nu} = \partial_\mu \psi \partial_\nu \psi - \frac{1}{2} g_{\mu\nu} \left(\partial \phi\right)^2 \tag{2}$$

$$\nabla^\mu \left(T_{\mu\nu} X^\nu\right) = T_{\mu\nu} {}^{(X)}\pi^{\mu\nu} \tag{3}$$

$$\Box \left(X\psi\right) = -2^{(X)}\pi^{\alpha\beta} \nabla_a \nabla_b \psi - 2 \left[2\nabla^{\alpha\,(X)}\pi_{\alpha\mu} - \nabla_\mu \left(tr^{(X)}\pi\right)\right] \nabla^\mu \psi$$

The wave equation $\Box_g \psi = 0$ on fixed black hole backgrounds

well-understood on Schwarzschild and Kerr and, in some circumstances, certain classes of perturbations thereof. Important insights for the decay mechanism of linear waves:

- redshift effect near the horizon ([Dafermos-Rodnianski] as multiplier and commutator)
- integrated decay estimate and the role of trapping (loss of derivatives and minimizing it, [Blue-Soffer], ...)
- improving and optimizing the decay-rates [Luk, Tataru]
- understanding the role of superradiance in the Kerr case and its relation to trapping [Daf.-Rod., Tataru-Tohaneanu, Andersson-Blue]
- avoiding the Morawetz vectorfield K [Dafermos-Rodnianski]

Spin 1 and Spin 2

Spin 1: Maxwell on fixed Schwarzschild background

$$\nabla^a F_{ab} = 0 \quad , \quad \nabla_{[a} F_{bc]} = 0 \tag{4}$$

This was resolved by [Blue] (decay for $l \geq 1$); main idea: spin reduction. A scalar component of the Maxwell field decouples and satisfies a wave equation (Regge-Wheeler for $s = 1$) for which decay estimates are obtained similarly to $\Box\psi = 0$. Decay for this component implies decay for the remaining components.

Spin 2: this talk; we will recognize the above mechanism, albeit in a different context.

Spin 2

The spin 2 case is very different in nature for the following reason. The equation

$$\nabla_S^a W_{abcd} = 0 \tag{5}$$

for W a Weyl field does not exhibit non-trivial dynamics if considered on a fixed Schwarzschild background. (There is an algebraic constraint on the curvature tensor, "Buchdahl constraint") Obviously, (5) is not the correct linearization of the Bianchi equation

$$\nabla^a \mathbf{W}_{abcd} = 0 \tag{6}$$

for spacetimes close to Schwarzschild. Since not all components of the curvature tensor decay

$$\left[\nabla_S^a + \epsilon \cdot \nabla^{(1)a}\right] \left(W^{(0)}_{abcd} + \epsilon \cdot W^{(1)}_{abcd}\right) \tag{7}$$

Hence we will see terms like $\rho \cdot$ (Ricci coefficients) on the right hand side. We forget about the linear problem and introduce the concept of **ultimately Schwarzschildean spacetimes**. This is a class of spacetimes satisfying $R_{\mu\nu} = 0$ and approaching the Schwarzschild metric for late times.

In this setting we have the dynamical variables $\mathfrak{R}$ (Ricci coefficients) and the curvature components of W. The equations are $(R_{\mu\nu} = 0)$

$$\nabla^a W_{abcd} = 0 \quad \text{and} \quad D\mathfrak{R} + \mathfrak{R}\mathfrak{R} = W \tag{8}$$

We introduce a null-frame $e_1, ..., e_4$ and decompose the equations, just as in the work of Christodoulou-Klainerman. I will present some of these equations later but stick to the schematic notation for as long as possible. $\boxed{\rho = W(e_3, e_4, e_3, e_4) = -\frac{2M}{r^3} \quad \text{, others vanish}}$

Fix regular coordinate system $(t^\star, r, \theta, \phi)$ + differentiable structure of Schwarzschild of mass M. At the heart of the definition are

$$\mathbb{I}^n \left[\mathfrak{R}\right] \left(\tilde{\mathcal{M}}\left(\tau_1, \tau_2\right)\right) = \sum_{i=0}^{n} \int_{\tilde{\mathcal{M}}(\tau_1,\tau_2)} \| D^i \left(\mathfrak{R} - \mathfrak{R}_{SS}\right) \|^2 \tag{9}$$

$$\mathbb{E}^n \left[\mathfrak{R}\right] \left(\tilde{\Sigma}_\tau\right) = \sum_{i=0}^{n} \int_{\tilde{\Sigma}_\tau} \| D^i \left(\mathfrak{R} - \mathfrak{R}_{SS}\right) \|^2 \tag{10}$$

Ultimately Schwarzschildean to order $n+1$ means that

$$\mathbb{D}^n \left[\mathfrak{R}\right] (\tau) = \mathbb{I}^n \left[\mathfrak{R}\right] \left(\tilde{\mathcal{M}}\left(\tau, \infty\right)\right) + \mathbb{E}^n \left[\mathfrak{R}\right] \left(\tilde{\Sigma}_\tau\right) \leq C$$

$$\mathbb{D}^{n-1} \left[\mathfrak{R}\right] (\tau) = \mathbb{I}^{n-1} \left[\mathfrak{R}\right] \left(\tilde{\mathcal{M}}\left(\tau, \infty\right)\right) + \mathbb{E}^{n-1} \left[\mathfrak{R}\right] \left(\tilde{\Sigma}_\tau\right) \leq \frac{C}{\tau}$$

$$\mathbb{D}^{n-3} \left[\mathfrak{R}\right] (\tau) = \mathbb{I}^{n-3} \left[\mathfrak{R}\right] \left(\tilde{\mathcal{M}}\left(\tau, \infty\right)\right) + \mathbb{E}^{n-3} \left[\mathfrak{R}\right] \left(\tilde{\Sigma}_\tau\right) \leq \frac{C}{\tau^{\frac{5}{2}}}$$

Why this hierarchy? The expectation is that assuming boundedness and integrated decay for n derivatives of $\mathfrak{R}$ we will be able to show boundedness for n-derivatives of curvature. But then, due to the trapping, we lose a derivative every time we win a power in t in the decay. We don't want to assume more on the deformation tensor then what we expect to get back from the bounds proven on curvature.

Theorem 1 (Boundedness)**.** *Assume* $(\mathcal{M}, g)$ *is ultimately Schwarzschildean to order* $n+1$ *for some* $n \geq 7$ *and that the vectorfield* $T = \partial_{t^\star}$ *is still null on the horizon. Then we have the estimate*

$$\sup_{\tau} \mathbb{E}^n \left[W\right] \left(\Sigma_\tau\right) < C \left(\mathbb{D}^n \left[\mathfrak{R}\right] + \mathbb{E}^n \left[W\right] \left(\Sigma_0\right)\right) \tag{11}$$

for a uniform constant C depending on M only.

In other words, assuming a non-degenerate integrated decay estimate for $n-1$ derivatives of curvature, we can show boundedness of the energy of n-derivatives of curvature.

Theorem 1 does not need construction of a Morawetz vectorfield X. It is based purely on estimates arising from the ultimately Killing field T and the redshift. This exploits an observation made by [Dafermos-Rodnianski] in the context of the wave equation on Schwarzschild.

Once we bring in integrated decay estimates (Theorem 2, integrated decay) we will be able to remove the assumption on T. Then we will also be able to control a *degenerate* (at $r = 3M$) spacetime norm for n derivatives. However, as a pay-off, one will have to impose an additional condition at the highest order of derivatives (to exclude stationary modes/ nearby Kerr solutions).

For now let us understand the proof of Theorem 1.

The general method is from [ChrKl] based on L^2 energy estimates from the Bel-Robinson tensor. Let $\mathcal{W}$ be a Weyl field satisfying $\nabla^\alpha \mathcal{W}_{\alpha\beta\gamma\delta} = \mathcal{J}_{\beta\gamma\delta}$.

$$Q\left[\mathcal{W}\right]_{\alpha\beta\gamma\delta} = \mathcal{W}_{\alpha\rho\gamma\sigma}\mathcal{W}_{\beta\ \delta}^{\ \rho\ \sigma} + {}^\star\mathcal{W}_{\alpha\rho\gamma\sigma}{}^\star\mathcal{W}_{\beta\ \delta}^{\ \rho\ \sigma}\,. \tag{12}$$

The tensor Q is symmetric and traceless and satisfies the divergence identity

$$D^\alpha\left(Q_{\alpha\beta\gamma\delta}X^\beta Y^\gamma Z^\delta\right) = K_1^{XYZ}\left[\mathcal{W}\right] + K_2^{XYZ}\left[\mathcal{W}\right] \tag{13}$$

$$K_1^{XYZ}\left[\mathcal{W}\right] = Q^{\alpha\beta\gamma\delta}\left({}^{(X)}\pi_{\alpha\beta}Y_\gamma Z_\delta + {}^{(Y)}\pi_{\alpha\beta}Z_\gamma X_\delta + {}^{(Z)}\pi_{\alpha\beta}X_\gamma Y_\delta\right)$$

$$K_2^{XYZ}\left[\mathcal{W}\right] = \left[\mathcal{W}_{\beta\ \delta}^{\ \mu\ \nu}\mathcal{J}_{\mu\gamma\nu} + \mathcal{W}_{\beta\ \gamma}^{\ \mu\ \nu}\mathcal{J}_{\mu\delta\nu} + {}^\star\mathcal{W}_{\beta\ \delta}^{\ \mu\ \nu}\mathcal{J}^\star_{\mu\gamma\nu} + {}^\star\mathcal{W}_{\beta\ \gamma}^{\ \mu\ \nu}\mathcal{J}^\star_{\mu\delta\nu}\right]X^\beta Y^\gamma Z^\delta$$

for any spacetime vectorfields X, Y, Z. The term $K_2^{XYZ}\left[W\right]$ vanishes in the case of the homogeneous Bianchi equations.

It is well-known that $Q(X, Y, Z, n) \geq 0$ holds for any future directed causal vectorfields X, Y, Z, n. If they are timelike, then $Q(X, Y, Z, n)$ in fact controls the sum of squares of all null components of $\mathcal{W}$. Note also that the right hand side and the left hand side of (13) both depend only on $\mathcal{W}$ and not derivatives therof.

We consider the inhomogeneous equation $\nabla^{\alpha} \mathcal{W}_{\alpha\beta\gamma\delta} = \mathcal{J}_{\beta\gamma\delta}$ directly because we will have to commute the equation with vectorfields to estimate higher derivatives, introducing an error on the right hand side:

$$D^{\alpha} \left(\widehat{\mathcal{L}}_X W \right)_{\alpha\beta\gamma\delta} = {}^{(X)}\pi \cdot DW + D^{(X)}\pi \cdot W \qquad (14)$$

Here $\widehat{\mathcal{L}}_X$ is a modified Lie-derivative which gives $\widehat{\mathcal{L}}_X W$ the algebraic properties of a Weyl-tensor. Clearly, deriving energies for $\widehat{\mathcal{L}}_T W$ will necessitate an analysis of the non-vanishing term $K_2^{XYZ}[W]$.

The vectorfield $T = \partial_{t^\star}$ is no longer Killing but only ultimately Killing, which in particular implies $t^{-\frac{3}{2}}$-decay of its deformation tensor. It is not hard to see

Lemma 1. *The error term* $K_1^{TTT}[\mathcal{W}]$ *satisfies*

$$\int_{\mathcal{M}(\tau_1,\tau_2)} K_1^{TTT}[\mathcal{W}] \leq \frac{\epsilon}{\sqrt{\tau_1}} \cdot \sup_{\tau \in (\tau_1,\tau_2)} \int_{\Sigma_\tau} |\mathcal{W}|^2 r^2 dr d\omega \qquad (15)$$

Lemma 2.

$$\int_{\Sigma_\tau} Q(T,T,T,n_\Sigma)\, d\mu_{\Sigma_\tau} \geq b \int_{\Sigma_\tau \cap \{r \geq r_Y - \frac{3}{4}(r_Y - 2M)\}} |\mathcal{W}|^2 r^2 dr d\omega - \epsilon \int_{\Sigma_\tau \cap \{r \leq r_Y\}} |\mathcal{W}|^2 r^2 dr d\omega \qquad (16)$$

The last term will be dealt with once we introduced the redshift. In applications, $\mathcal{W} = \widehat{\mathcal{L}}_T^k W$. Note that in the case $k = 0$, the norm on the right hand side does not decay.

It remains to control $\int_{\mathcal{M}(\tau_1,\tau_2)} K_2^{XYZ}[W]$. This is the hardest part. Recall

$$D^\alpha \left(\widehat{\mathcal{L}}_X W\right)_{\alpha\beta\gamma\delta} = {}^{(X)}\pi \cdot DW + D^{(X)}\pi \cdot W \tag{17}$$

and hence

$$\int_{\mathcal{M}(\tau_1,\tau_2)} \left(\widehat{\mathcal{L}}_T W\right) D\pi \cdot \rho \tag{18}$$

is not a cubic error-term! The situation doesn't improve upon taking derivatives: $\int_{\mathcal{M}(\tau_1,\tau_2)} \left(\widehat{\mathcal{L}}_T^n W\right) \widehat{\mathcal{L}}_T^{n-1} D\pi \cdot \rho$

Because we don't have a (non-degenerate) integrated decay estimate available, we have to put the higest derivative term L^2 in space and hence need a strong decay estimate for the deformation tensor. However, the assumptions of ultimately Schwarzschildean only provide *boundedness* of the corresponding spacetime integral

$$\int_{\mathcal{M}(\tau_1,\tau_2)} |\widehat{\mathcal{L}}_T^{n-1} D\pi|^2 < \infty \tag{19}$$

The resolution of this exploits the coupled character of the problem. Very schematically one can think

$$\int_{\mathcal{M}(\tau_1,\tau_2)} \left(\widehat{\mathcal{L}}_T^{n} W\right) \widehat{\mathcal{L}}_T^{n-1} D\pi \cdot \rho = \int_{\mathcal{M}(\tau_1,\tau_2)} \left(\widehat{\mathcal{L}}_T^{n} W\right) \widehat{\mathcal{L}}_T^{n-1} W \cdot \rho \tag{20}$$

by the null-structure equations, and then integrate by parts to obtain lower order boundary terms and a cubic error-term since the T derivative falls on ρ.

In reality, there are dozens of contractions to be taken to isolate all terms proportional to ρ and many null-structure equations enter. The key is a null-decomposition of the error-terms. In summary,

$$\left| \int_{\mathcal{M}(\tau_1,\tau_2)} \left(\widehat{\mathcal{L}}_T^n W \right) \widehat{\mathcal{L}}_T^{n-1} D\pi \cdot \rho \right| \leq B \left(\mathbb{D}^n \left[\mathfrak{R} \right] + \mathbb{D}^{n-1} \left[W \right] \right) < \infty \quad (21)$$

The other error-terms are cubic and cause no problem if sufficiently many derivatives are taken.

This almost proves Theorem 1, except that we are only estimating the T-derivatives and that we still have a wrong-signed term close to the horizon. The former problem can be dealt with elliptic estimates in the interior, which estimate the remaining derivatives form the T derivative (write Bianchi equations as *div-curl* system on the Σ-slices with T derivative on the right hand side).

The redshift : There is a timelike vectorfield $N = T + Y$...

Proposition 1. *The N-boundary-term satisfies*

$$Q\left(N, N, N, n_{\Sigma}^{\mu}\right) \geq b\left(|\underline{\alpha}|^2 + |\alpha|^2 + |\underline{\beta}|^2 + |\beta|^2 + \left(\rho^2 + \sigma^2\right)\right). \qquad (22)$$

everywhere on the black hole exterior. The quantity

$$K_1^{NNN}\left(\mathcal{W}\right) = Q_{abcd}\left({}^{(T+Y)}\pi\right)^{ab}\left(T+Y\right)^c\left(T+Y\right)^d \qquad (23)$$

satisfies

$$K_1^{NNN}\left(\mathcal{W}\right) \geq b \cdot Q\left(N, N, N, n_{\Sigma}^{\mu}\right) \qquad \text{for } r \leq r_Y$$

$$\left|K_1^{NNN}\left(\mathcal{W}\right)\right| \leq B \cdot Q\left(T, T, T, n_{\Sigma}\right) \qquad \text{for } r_Y \leq r \leq r_Y + \frac{r_Y - 2M}{2}$$

$$K_1^{NNN}\left(\mathcal{W}\right) = K_1^{TTT}\left(\mathcal{W}\right) \qquad \text{for } r \geq r_Y + \frac{r_Y - 2M}{2}$$

Actually, in view of the fact that we also have to commute with N near the horizon, it turns out that it is more convenient to work directly with the (renormalized) Bianchi equations and multipliers. Either way, the result is

Proposition 2 (Estimates close to the horizon)**.**

$$\mathbb{E}^n_{r\leq r_Y}\left[W\right]\left(\Sigma_{\tau_2}\right)+\mathbb{E}^n\left[W\right]\left(\mathcal{H}\left(\tau_1,\tau_2\right)\right)+\mathbb{I}^{n,deg}_{r\leq r_Y}\left[W\right]\left(\mathcal{M}\left(\tau_1,\tau_2\right)\right)\leq$$
$$2\cdot\mathbb{E}^n_{r\leq r_Y}\left[W\right]\left(\Sigma_{\tau_1}\right)+B\cdot\mathbb{I}^{n,deg}_{|r-r_Y|\leq\frac{r_Y-2M}{2}}\left[W\right]\left(\mathcal{M}\left(\tau_1,\tau_2\right)\right)+\mathbb{D}^n\left[\mathfrak{R}\right]$$

Adding enough of the T estimate to this (+using the elliptic estimates) we derive Theorem 1 following the argument for the wave equation.

With this, we turn to part II of the talk: **Decay**.

We would like to prove an integrated decay estimate. Expectation: Degenerate at the highest order and, ideally we can "improve" the constant assumed one order lower.

The obvious idea is to try a vectorfield $X = f(r)\,\partial_r$ with a bounded f as for the wave equation. One easily sees that one cannot obtain global positivity of a spacetime integral of all curvature components. This is already clear from the fact that ρ does not decay.

$$K_1^{XTT} = \frac{(1-\mu)^2}{4} f_{,r} \left[\frac{1}{2}(1-\mu)^2 |\underline{\alpha}|^2 + \frac{1}{2}\frac{|\alpha|^2}{(1-\mu)^2} - \frac{1-\mu}{2}\left(\rho^2+\sigma^2\right)\right]$$
$$+ f\left(\frac{1}{r} - \frac{3}{2}\frac{\mu}{r}\right)\left[(1-\mu)^2 |\underline{\beta}|^2 + |\beta|^2 + 2(1-\mu)\left(\rho^2+\sigma^2\right)\right].$$

Proposition 3. *Let $\mathcal{W}$ be a Weyl-field satisfying the inhomogeneous Bianchi equations. We have the integrated decay estimates*

$$I^{deg}\left[\mathcal{W}\right]\left(\tilde{\mathcal{M}}\left(\tau_1,\tau_2\right)\right) \leq B \int dt^{\star} \int_{r_0}^{R} dr\, r^2 \int d\omega \frac{\left(r-3M\right)^2}{r^3}\left[|\rho|^2+|\sigma|^2\right]$$
$$+B \cdot E\left[\mathcal{W}\right]\left(\tilde{\Sigma}_{\tau_2}, \tilde{\Sigma}_{\tau_1}, \mathcal{H}\left(\tau_1,\tau_2\right)\right) + errors$$

$$I^{nondeg}\left[\mathcal{W}\right]\left(\tilde{\mathcal{M}}\left(\tau_1,\tau_2\right)\right) \leq B \int dt^{\star} \int_{r_0}^{R} drr^2 \int d\omega \left[|\rho|^2+|\sigma|^2\right]$$
$$+B \cdot E\left[\mathcal{W}\right]\left(\tilde{\Sigma}_{\tau_2}, \tilde{\Sigma}_{\tau_1}, \mathcal{H}\left(\tau_1,\tau_2\right)\right) + errors$$

In other words, one needs to establish decay of ρ and σ to obtain integrated decay for everyone else ("spin reduction"). Proof requires Maxwell pseudotensor $\mathcal{F}_{ab} = r \cdot \mathcal{W}\left(e_a, e_b, e_3, e_4\right)$, which satisfies an inhomogeneous Maxwell equation.

How does one obtain decay of ρ and σ? Let us focus on ρ. It turns out the natural quantity to consider is

$$\phi_k = r^3 \left[\rho\left(\widehat{\mathcal{L}}_T^k W\right) + \frac{2M}{r^3}\delta_0^k\right] \tag{24}$$

which – perhaps not surprisingly – satisfies the Regge-Wheeler equation for $s=2$:

$$-\frac{1}{1-\mu}\partial_u\partial_v\phi_k + \frac{1}{r^2}\Delta_{S^2}\phi_k + \frac{6M}{r^3}\phi_k = r^3\left(\mathcal{F}^k + \underline{\mathcal{F}}^k\right)$$

where $\mathcal{F}^k$ are of the form $\mathcal{D}^{k+1}\left[(\text{Ricci-coefficient})\cdot(W)\right]$. If the right hand side were zero: Decay for $l \geq 2$ in Schwarzschild [Blue].

Associated energy:

$$E\left[\mathcal{D}\phi_k\right]\left(\tilde{\Sigma}_\tau\right) = \int_{\Sigma, r\leq R} dr d\omega \|\mathcal{D}\phi_k\|^2 + \int_{N_{out}(\tau,R)} dv d\omega \left(|\not{D}_4\phi_k|^2 + |\not{\nabla}\phi_k|^2\right)$$

We immediately see the potential problem: The error has picked up a large weight in r!

The conclusion is that one cannot prove decay for ϕ_k unless one understands at the same time the characteristic decay in r of the individual curvature components as in the stability of Minkowski space. This will be discussed in the last part of the talk.

One method to go about this is using the K-vectorfield. However, we will adapt the more flexible method recently developed by [DafRod] to the spin2 setting.

To see the ideas more clearly, we will from now assume we are in Minkowski-space.

The Bianchi equations for the components α and β take the form

$$2\partial_u \alpha - \frac{1}{r}\alpha = -2\not{\mathcal{D}}_2^\star \beta + \text{l.o.t.} \tag{25}$$

$$2\partial_v \beta + \frac{4}{r}\beta = \not{d}\text{iv}\,\alpha + \text{l.o.t.} \tag{26}$$

Multiplying the first by $r^{p_1}\alpha$ and using that $\not{\mathcal{D}}_2^\star$ is the adjoint of $\not{d}\text{iv}$ on the 2-spheres, we obtain

$$\begin{aligned} &\partial_u \left(r^{p_1}\|\alpha\|^2\right) + 2\partial_v \left(r^{p_1}\|\beta\|^2\right) + \|\alpha\|^2 r^{p_1-1}\left(-p_1 \cdot \partial_u r - 1\right) \\ &\qquad + \|\beta\|^2 r^{p_1-1}\left(8 - 2p_1 \partial_v r\right) = \text{error} + \text{tot. div on } S^2_{t,r}\,. \end{aligned} \tag{27}$$

Upon integration in a characteristic region, the first two terms will generate positive future boundary terms. The remaining terms will both be positive as long as $2 < p_1 < 8$.

In other words, provided we can control the error-term arising on the timelike boundary of the region (which can be done, provided an integrated decay estimate is available) and the error-terms on the right hand side, we obtain an estimate for strongly r-weighted boundary *and* spacetime terms for the components α and β.

Considering the next pair of Bianchi equations, we will obtain a similar estimate for the "pair" $(\beta, [\rho, \sigma])$ with the condition $4 < p_2 < 6$. Actually, the lower bound is irrelevant as long as $p_2 \leq p_1$. (it is the condition for the β term, which we control from the previous estimate).

We step down to $\underline{\alpha}$ and add everything up to obtain

$$\int_{N_{out}\left(S^2_{\tau_2,R}\right)} dv d\omega \left[r^{p_1}\|\alpha\|^2 + r^{p_2}\|\beta\|^2 + r^{p_3}\|\widehat{\rho},\sigma\|^2 + r^{p_4}\|\underline{\beta}\|^2\right]$$

$$+\int_{\tilde{\mathcal{M}}(\tau_1,\tau_2)} dt^\star dr d\omega \left[r^{p_1-1}\|\alpha\|^2 + r^{p_1-1}\|\beta\|^2\right.$$

$$\left.+r^{p_2-1}\|\widehat{\rho},\sigma\|^2 + r^{p_3-1}\|\underline{\beta}\|^2 + r^{p_4-1}\|\underline{\alpha}\|^2\right]$$

$$\leq \int_{N_{out}\left(S^2_{\tau_1,R}\right)} dv d\omega \left[r^{p_1}\|\alpha\|^2 + r^{p_2}\|\beta\|^2 + r^{p_3}\|\widehat{\rho},\sigma\|^2 + r^{p_4}\|\underline{\beta}\|^2\right]$$

$$+\mathbb{I}^0\left[\hat{W}\right]\left(\mathcal{M}\left(\tau_1,\tau_2\right)\cap\{r\leq R+1\}\right) + \text{errors}$$

Notion of admissible tuples (for Schwarzschild there are constraints from the non-linear errors!) + Generalization to higher derivatives (improved p-weights for commutation by 4-derivatives).

The pigeonhole principle is much cleaner in this case. Characteristic decay in u for $\underline{\alpha} r$, $\underline{\beta} r^2$, etc. is easily obtained.

Commutation by 3 and 4-derivatives + Bianchi equations is sufficient to estimate all derivatives (however, for optimal r-weights we need the angular momentum vectorfields).

We now couple the X estimate for the Bel-Robinson tensor with the estimate for ρ (and σ) to obtain (for "$l \geq 2$")

$$\mathbb{I}^{n,deg}\left[W\right]\left(\mathcal{M}\left(\tau_1,\tau_2\right)\right) \leq B \cdot \sup_{\tau} \mathbb{E}^n_{wgh}\left[W\right]\left(\Sigma_\tau\right) + \epsilon \cdot \mathbb{I}^n_{wgh}\left[W\right]\left(\mathcal{D}^{\tau_2}_{\tau_1}\right) + \mathbb{D}^n\left[\mathfrak{R}\right] .$$

which gets coupled to the estimate at infinity to produce

Theorem 2 (Local r-weighted energy decay)**.** *Assume* $(\mathcal{M}, g)$ *is ultimately Schwarzschildean to order* $n+1$ *for some* $n \geq 7$*. Let the "*$l \geq 2$*-assumption" (next slide) hold. Then the estimates*

$$\mathbb{E}^{n}_{wgh}\left[W\right]\left(\tilde{\Sigma}_{\tau_2}\right) + \mathbb{I}^{n,deg}_{wgh}\left[W\right]\left(\tilde{\mathcal{M}}\left(\tau_1, \tau_2\right)\right) \leq B \cdot \mathbb{E}^{n}_{wgh}\left[W\right]\left(\tilde{\Sigma}_{\tau_1}\right) + B \cdot \mathbb{D}^{n}\left[\mathfrak{R}\right]\left(\tau_1, \tau_2\right) \qquad (28)$$

and, for any $\lambda > 0$,

$$\mathbb{I}^{n-1,nondeg}_{wgh}\left[W\right]\left(\tilde{\mathcal{M}}\left(\tau_1, \tau_2\right)\right) \leq B \cdot \mathbb{E}^{n}_{wgh}\left[W\right]\left(\tilde{\Sigma}_{\tau_1}\right) + \lambda \cdot \mathbb{D}^{n}\left[\mathfrak{R}\right]\left(\tau_1, \tau_2\right) + B_{\lambda} \cdot \mathbb{D}^{n-1}\left[\mathfrak{R}\right]\left(\tau_1, \tau_2\right) \qquad (29)$$

hold.

This provides a stronger (r-weighted) boundedness statement. Note also that the right hand side of (29) is small for late enough τ.

The $l \geq 2$ assumption

The quantities $\phi_i = r^3\rho\left(\widehat{\mathcal{L}}_T^i W\right) + 2M\delta_0^i$ and $\psi_i = r^3\sigma\left(\widehat{\mathcal{L}}_T^i W\right)$ each satisfy the Poincare inequality

$$r^2 \int_{S^2} |\nabla\!\!\!/ \phi_i|^2 r^2 dA_{S^2} \geq c_\rho \int_{S^2} |\phi_i|^2 r^2 dA_{S^2} \tag{30}$$

for a constant $c_\rho = \frac{23}{4} < 6$ and for all $i = 0, ..., n$.

If the metric was exactly Schwarzschild, the above would be an identity with $c_\rho = l\,(l+1)$ and the assumption would in particular imply the absence of the $l = 1$ and the $l = 0$-mode in the quantities ϕ_i, ψ_i.