Black Hole Topology in Higher Dimensions

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  In an asymptotically flat spacetime, every causal curve in the domain of outer communications with endpoints at $\mathscr{I}$ can be deformed to a curve entirely in $\mathscr{I}$.

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In 4 dimensions, these theorems specify the horizon topology of a stationary, asymptotically flat black hole uniquely.

Horizon Topology in 5-dimensions

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Topology Theorem in $5D$ [Hollands-H.-Ishibashi]

A horizon cross section of a stationary, analytic, asymptotically flat vacuum black hole in 5 dimensions has topology

$$\mathcal{H} \cong \begin{cases} \# l \cdot \left(S^1 \times S^2\right) \, \# L(p_1, q_1) \, \# \dots \# L(p_k, q_k) \\ \\ S^3/\Gamma \, , \text{ where } \Gamma \text{ is a finite subgroup of } SO(4) \end{cases}$$

where $k, l, p_i, q_i \in \mathbb{N}$ and $L(p_i, q_i)$ are Lens spaces (i.e. certain quotients of S^3 by $\mathbb{Z}_{p_i}$).

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Note: No connected sums of spherical manifolds allowed.

The proof combines the following ingredients:

- Positive curvature theorem
- Topological censorship
- Rigidity Theorem
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Orbit type	Isotropy group	Properties
Principal orbits	id	dense and open in $\mathcal{H}$
Fixed points	$U(1)$	lie on l boundary circles in $\hat{\mathcal{H}}$
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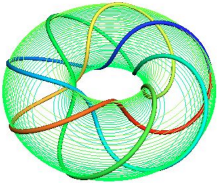
Each exceptional orbit is surrounded in $\mathcal{H}$ by a standard fibered torus with winding numbers (p_i, q_i).



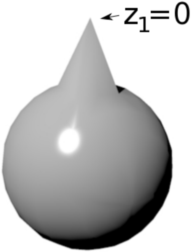
Example ($M = S^3 = 3-$Sphere, $G = U(1)$)

$$S^3 = \left\{ (z_1, z_2) \in \mathbb{C}^2 \,\middle|\, |z_1|^2 + |z_2|^2 = 1 \right\}$$

Let $e^{i\alpha} \in U(1)$ act by

$$(z_1, z_2) \to \left(e^{i\alpha} z_1, e^{ip\alpha} z_2\right)$$

Then $\hat{M} = M/G$ is a two-sphere with an orbifold point.



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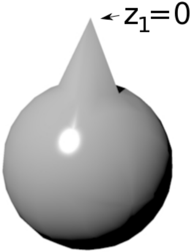

Theorem [Seifert, Orlik, Raymond]

A closed, connected, smooth and oriented 3-manifold with $U(1)$-action is determined up to equivariant diffeomorphisms by the decoration data

$$\{g; l; (p_1, q_1), \ldots, (p_k, q_k)\}$$

where g is the genus of $\hat{\mathcal{H}}$, l the number of disconnected boundary components and (p_i, q_i) the winding numbers introduced above.

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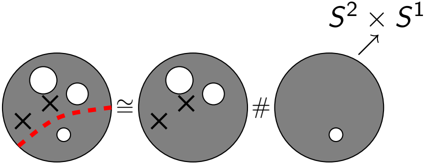

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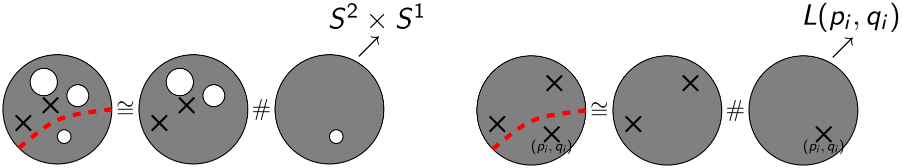

$\mathcal{H} \cong \#(l-1) \cdot (S^2 \times S^1)$ $\qquad$ $\#L(p_1, q_1)\# \dots \#L(p_k, q_k)$

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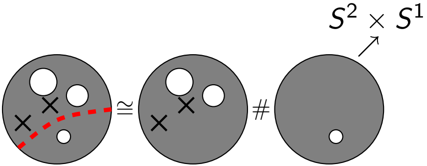

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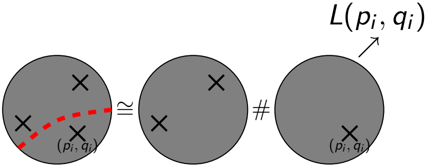

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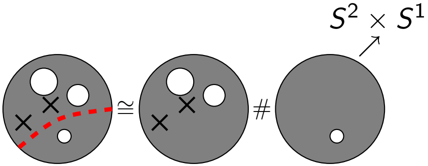

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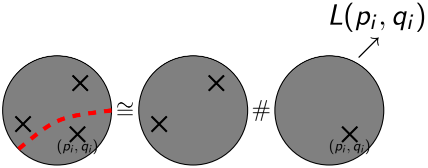

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$\partial\hat{\mathcal{H}} = \emptyset$: Seifert manifolds.

Aim: Show that $\chi_{\mathrm{Orb}}(\hat{\mathcal{H}}) = 2 - 2g - \sum_i \left(1 - \frac{1}{p_i}\right) > 0$, then by standard results $\mathcal{H} \cong S^3/\Gamma$

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$$\tilde{\gamma} = \mathrm{e}^{\nu}(d\varphi + \omega_i dx^i)^2 + \mathrm{e}^{-\nu} h_{ij} dx^i dx^j$$

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$$\mathrm{e}^{-\nu} R[\tilde{\gamma}] = R[h] - \frac{1}{4}\mathrm{e}^{2\nu}\mathcal{F}_{ij}\mathcal{F}^{ij} - \frac{1}{2}(\partial_i \nu)\partial^i \nu$$

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  - for small r the metric h takes the form $h \sim dr^2 + r^2 dy^2$ where y is $2\pi/p_i$ periodic, so computing the integrals in the limit $r \to 0$ yields the desired result

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Theorem [Hollands-H.-Ishibashi]

For a spacetime as specified above, the domain of outer communications has topology $M \cong \Sigma \times \mathbb{R}$, where

$$\Sigma \cong \left(\mathbb{R}^4 \# n \cdot (S^2 \times S^2) \# n' \cdot (\pm CP^2) \right) \setminus B$$

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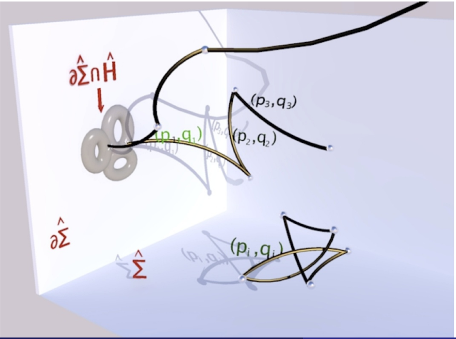

The proof again relies on an analysis of the orbit space $\hat{\Sigma}$, which is a three dimensional manifold with boundaries corresponding to fixed points and polyhedral arcs corresponding to exceptional orbits.

What happens if we assume an additional killing field?

Topology Theorem [Hollands, Yazadjiev]

For a D-dimensional stationary black hole with $(D-3)$ commuting Killing vector fields generating $U(1)^{D-3}$

$$\mathcal{H} \cong \begin{cases} S^3 \times T^{D-5} \\ S^2 \times T^{D-4} \\ L(p,q) \times T^{D-4} \end{cases}$$

Restrictions on the d.o.c. are the same as before, but now Einstein's equations can be formulated as a non-linear sigma model.

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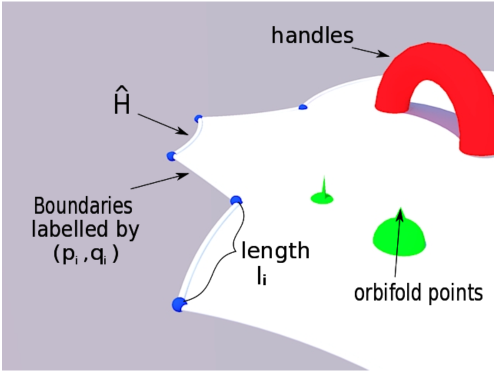

Handles and orbifold points are excluded by topological censorship.

Conclusions and Outlook

- Analysis of orbit space structure gives restrictions on black hole horizon topologies in higher dimensions
- The domain of outer communication can be analyzed using similar methods
- Further restrictions due to the assumption of additional symmetries can be determined

Future Topics:

- Extension to matter fields
- Generalization to higher (arbitrary) dimension
- Ultimate goal: Find parameters that uniquely classify black hole solutions in higher dimensions