Geodesity of WANDs in higher dimensions

Mark Durkee

DAMTP, University of Cambridge
M.N.Durkee@damtp.cam.ac.uk

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Introduction

The CMPP formalism is a mathematical method for studying general relativity in arbitrary dimension $d \geq 4$.

- For some spacetimes (including e.g. Myers-Perry black holes), there is a preferred choice of frame, based around a ‘multiple WAND’, which simplifies things.
- Generalization of Petrov classification and Newman-Penrose formalism to higher dimensions.
- CMPP = Coley, Milson, Pravda, Pravdova (2004)

Main result of the talk concerns geodesity of multiple WANDs.

Possible Applications

Doing this in higher dimensions is a moderately new idea, some possible applications are:

- Finding new solutions to GR in higher dimensions *(very successful in $d = 4$, not yet for $d > 4$)*
- Classifying known solutions? *(some limited progress)*
- Perturbations? - Teukolsky equation comes from Newman-Penrose approach. *(recall Harvey's talk this morning)*

Outline

Null frames

Consider a null frame $\{e_0, e_1, \ldots, e_{d-1}\}$, with e_0, e_1 null, and e_i spacelike ($i = 2, ..., d-1$). Have

$$\eta_{ab} = e_a . e_b = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix} .$$

Write $\ell = e_0 = e^1$, $n = e_1 = e^0$, $m_i = e_i = e^i$.

Introduction to CMPP Formalism

Coley, Milson, Pravda and Pravdova (2004) started the development of a formalism for doing *d*-dimensional GR in null frames (generalizing Newman-Penrose formalism/Petrov classification in 4D).

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Take frame basis components of any tensors T:

$$T_{ab...c} \equiv T(e_a, e_b, \dots, e_c).$$

Each object $T_{ab...c}$ is now a spacetime *scalar*, so for example,

$$\nabla_\mu T_{ab...c} = \partial_\mu T_{ab...c}.$$

The Lorentz group $SO(1, d-1)$ acts to change basis.

Action of the Lorentz group

We divide up the action of the Lorentz group on the basis vectors as follows:

Spins $SO(D-2)$ rotations of the spatial basis vectors m_i.

Boosts Under a local Lorentz boost we get

$$\ell \to \lambda \ell, \qquad n \to \lambda^{-1} n, \qquad m_i \to m_i,$$

and we say that ℓ, n and m_i have *boost weights* $+1$, -1 and 0 respectively.

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Null Rotations Rotations of one of the null basis vectors about the other, for example a null rotation about n takes the form

$$\ell \to \ell - z_i n - \frac{1}{2} z^2 \ell, \qquad n \to n, \qquad m_i \to m_i + z_i n$$

for some z_i.

Classification by boost weight

We can use this to make some definitions that turn out to be useful:

- The idea is that we classify components of tensors by their *boost weights b*: $T_{abcd} \mapsto \lambda^b T_{abcd}$.
- Useful to apply this to the Weyl tensor $C_{\mu\nu\rho\sigma}$.

  (Recall: Weyl tensor is totally traceless part of Riemann curvature tensor).
- Just need to count the number of 0s and 1s in the indices to find boost weight of a component, as follows...

Classification of the Weyl tensor

What is the highest boost weight component?
(recall symmetries $C_{abcd} = C_{cdab} = C_{[ab]cd} = C_{ab[cd]}$ and $C_{a[bcd]} = 0$)

Boost Weight +2 $\Omega_{ij} \equiv C_{0i0j}$

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Boost Weight +1 $\Psi_{ijk} \equiv C_{0ijk}, \Psi_i \equiv C_{010i}$

Boost Weight 0 $\Phi_{ijkl} \equiv C_{ijkl}, \Phi_{ij} \equiv C_{0i1j},$
$2\Phi^{\mathrm{A}}_{ij} \equiv C_{01ij}, \Phi \equiv C_{0101}$

Boost Weight -1 $\Psi'_{ijk} \equiv C_{1ijk}, \Psi'_i \equiv C_{101i}$

Boost Weight -2 $\Omega'_{ij} \equiv C_{1i1j}$

Recall Petrov classification in 4D

Those familiar with the Petrov classification will realise that all these components can be written in terms of complex scalars Ψ_A in 4 dimensions.

Boost Weight +2 $\Psi_0 \sim \Omega_{ij}$

Boost Weight +1 $\Psi_1 \sim \Psi_{ijk}, \Psi_i$

Boost Weight 0 $\Psi_2 \sim \Phi_{ijkl}, \Phi_{ij}$

Boost Weight -1 $\Psi_3 \sim \Psi'_{ijk}, \Psi'_i$

Boost Weight -2 $\Psi_4 \sim \Omega'_{ij}$

We can't do this in general dimension - too many independent compts.

Definition of a WAND

Definition

- A null vector field ℓ is a *Weyl-aligned null direction (WAND)* iff all boost weight +2 components of the Weyl tensor vanish in a frame containing ℓ.
  (In 4D this is equivalent to being a PND)
- ℓ is a *multiple WAND* iff all boost weight +2 and +1 components vanish.
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Definition

A spacetime is *algebraically special* if it admits a multiple WAND.

Existence of WANDs

The natural first question is do WANDs always exist? In conformally flat spacetimes ($C_{abcd} = 0$), any null vector field is a WAND, otherwise:

- In $D = 4$, yes. Any spacetime admits exactly 4 WANDs, some possibly repeated.

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- In $D = 4$, yes. Any spacetime admits exactly 4 WANDs, some possibly repeated.
- In $D > 4$, no. An arbitrary spacetime might admit no WANDs, a finite number of WANDs, or even a continuous family.
- *NB:Existence is a local property in general, but for analytic spacetimes can extend this globally (so in a smooth, non-analytic spacetime, everything I say is valid in some open neighbourhood of any point in a spacetime).*

Algebraic Types

Given a spacetime, we look to pick ℓ so that as many high boost weight components of ℓ vanish as possible. Different algebraic types are defined based on which components of C_{abcd} vanish in this chosen frame:

Type	b					
G	+2	+1	0	-1	-2	General
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D			0			n also mWAND
O						Conformally flat

Algebraic Types of Black Holes

Generically, spacetimes are not algebraically special, but many important metrics are, for example:

- Schwarzchild: Type D
- Kerr/Myers-Perry-((A)dS): Type D
- C-metric (known in 4d only): Type D
- Kerr-Schild spacetimes: Type II or more special Ortaggio et al. '09

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This looks like it might be a general statement about black holes...but:

- Singly-Spinning Black Ring:
  - Type II on the horizon
  - Type I or G elsewhere.

Constructing new solutions?

The Kerr metric was discovered by looking for an axisymmetric, algebraically special solution of the vacuum Einstein equations. Can we find any interesting new solutions in higher-dimensions like this?

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The Kerr metric was discovered by looking for an axisymmetric, algebraically special solution of the vacuum Einstein equations. Can we find any interesting new solutions in higher-dimensions like this?

- Answer: Not yet...
- Lots more scope for trying.
- Think of this as a simplifying assumption...might or might not make things tractable.
- Potentially useful for AdS-CFT as studying solutions with AdS asymptotics seems no more difficult than asymptotically flat (c.f. inverse scattering techniques where this is definitely not true).

Aside: Different 4D approaches

In 4D there are various equivalent ways of describing 'Weyl-alignment':

Bivectors $\equiv$ 2-forms

We can think of the Weyl tensor as a linear map from bivectors to bivectors. Look for eigenvectors of this map. Conceptually easy, not all that useful for calculations.

2-spinors

Use relationship between $SU(2)$ and $SO(1,3)$ to write everything in terms of complex 2-spinors. This simplifies everything lots. But can't be done in general dimension.

Vectors (real/complex)

We're using the real version. Easiest to formulate in arbitrary dimension, but not most straightforward for 4-dimensions.

Describing optical properties

Consider null congruence defined by ℓ. Define

$$\rho_{ij} \equiv m_i.[(m_j.\nabla)\ell].$$

Then we can split up ρ_{ij} into:

- An antisymmetric part ω_{ij}, describing **twist.**
- A symmetric, traceless part σ_{ij} describing **shear.**
- A trace, $\frac{\rho}{d-2}\delta_{ij}$, describing **expansion.**

The congruence is **geodesic** if and only if

$$\kappa_i \equiv m_i.[(\ell.\nabla)\ell] = 0.$$

Four dimensions

The conditions for a null vector field to be a multiple WAND are very complicated to check in practice, as the Weyl tensor is complicated. It would be good if there was a simpler way of checking whether something was a multiple WAND.

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In 4 dimensions an important result is:

Theorem (Goldberg + Sachs 1962)

In a 4D Einstein spacetime, a null congruence is a multiple WAND iff it is geodesic and shearfree.

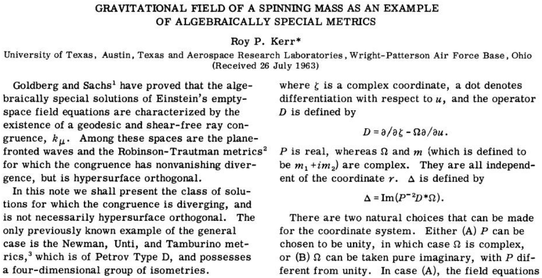
(An Einstein spacetime is a solution of the vacuum Einstein equations $R_{ab} = \Lambda g_{ab}$.)

In notation of last slide:

$$\ell \text{ multiple WAND} \Leftrightarrow \sigma_{ij} = 0 = \kappa_i .$$

Finding the Kerr solution

This was genuinely useful for constructing new solutions in 4D, in particular Kerr (1963) begins:



GRAVITATIONAL FIELD OF A SPINNING MASS AS AN EXAMPLE
OF ALGEBRAICALLY SPECIAL METRICS

Roy P. Kerr*

University of Texas, Austin, Texas and Aerospace Research Laboratories, Wright-Patterson Air Force Base, Ohio
(Received 26 July 1963)

Goldberg and Sachs[1] have proved that the algebraically special solutions of Einstein's empty-space field equations are characterized by the existence of a geodesic and shear-free ray congruence, k_μ. Among these spaces are the plane-fronted waves and the Robinson-Trautman metrics[2] for which the congruence has nonvanishing divergence, but is hypersurface orthogonal.

In this note we shall present the class of solutions for which the congruence is diverging, and is not necessarily hypersurface orthogonal. The only previously known example of the general case is the Newman, Unti, and Tamburino metrics,[3] which is of Petrov Type D, and possesses a four-dimensional group of isometries.

where ζ is a complex coordinate, a dot denotes differentiation with respect to u, and the operator D is defined by

$$D = \partial/\partial\zeta - \Omega\partial/\partial u.$$

P is real, whereas Ω and m (which is defined to be $m_1 + im_2$) are complex. They are all independent of the coordinate r. Δ is defined by

$$\Delta = \mathrm{Im}(P^{-2}D^*\Omega).$$

There are two natural choices that can be made for the coordinate system. Either (A) P can be chosen to be unity, in which case Ω is complex, or (B) Ω can be taken pure imaginary, with P different from unity. In case (A), the field equations

It's an extra simplification that can be applied to an ansatz for an algebraically special metric, without additional loss of generality.

Higher dimensions

The obvious generalization of the Goldberg-Sachs theorem fails for all $D > 4$:

- Some spacetimes admit non-geodesic multiple WANDs.
- Multiple WANDs, whether geodesic or not, can be shearing.

Is there a different generalization, that is some condition on the optics of ℓ that is equivalent to the algebraically special condition, reducing to “geodesic and shearfree” in 4D?

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Is there a different generalization, that is some condition on the optics of ℓ that is equivalent to the algebraically special condition, reducing to “geodesic and shearfree” in 4D?

Answer: We don’t know yet...

Geodesity in higher-dimensions

The generalization of the geodesic part is now well understood:

Theorem *(MND + Reall 2009)*

An Einstein spacetime admits a multiple WAND if, and only if, it admits a geodesic multiple WAND.

So given any algebraically spacetime, this allows us to work in the frame of a geodesic multiple WAND without loss of generality.

Sketch proof of geodesity theorem (1)

Previously, it was known that there were some significant restrictions on the form of the boost weight 0 components of the Weyl tensor when a non-geodesic multiple WAND exists:

Lemma (*Pravda et al 2007, MND 2009)*

In an Einstein spacetime that is not conformally flat, a multiple WAND ℓ is always geodesic if any of the following conditions on boost weight 0 components of the Weyl tensor hold:

1. Φ^{A}_{ij} *is non-vanishing.*
2. *None of the eigenvalues of* Φ^{S}_{ij} *are* $-\Phi$.
3. Φ_{ijkl} *vanishes identically.*

Proof of lemma

Proof.

Contracting the differential Bianchi identity $\nabla_{[\mu}C_{\nu\rho]\sigma\tau} = 0$ with vectors $m_i{}^{\mu}\ell^{\nu}m_j{}^{\rho}m_k{}^{\sigma}\ell^{\tau}$ gives

$$\Phi_{ij}\kappa_k - \Phi_{ik}\kappa_j + 2\Phi^{\mathrm{A}}_{kj}\kappa_i - \Phi_{iljk}\kappa_l = 0.$$

Taking various contractions of this with $\boldsymbol{\kappa}$ implies the equations

$$\Phi^{\mathrm{S}}_{ij}\kappa_i = -\Phi\kappa_j, \quad \Phi^{\mathrm{A}}_{ij}\kappa_i = 0, \quad \text{and} \quad |\boldsymbol{\kappa}|^2\Phi^{\mathrm{A}}_{ij} = 0,$$

which imply parts (i) and (ii) of the lemma. In case (iii), either $\Phi^{\mathrm{A}} \neq 0$ (so we're back in case (i)), or all boost weight 0 components of the Weyl tensor vanish, i.e. the spacetime is Type III or more, and by Pravda et al (2004) it is known that this means all multiple WANDs are geodesic.

☐

Sketch proof of geodesity theorem (2)

It turns out that this gives amazingly strong restrictions on the spacetimes that can admit non-geodesic multiple WANDs:

Lemma (*MND + Reall 2009*)

An Einstein spacetime admitting a non-geodesic multiple WAND is either Type D or conformally flat.

Recall that a Type D spacetime is one where we can pick both ℓ and n to be multiple WANDs.

Sketch proof of lemma

Proof.

- Write out the 16 Bianchi equations in the frame of a multiple WAND with $\Phi^{A}_{ij} = 0$ (so can work in a frame where Φ_{ij} is diagonal).
- By taking various components/contractions/linear combinations of them one can prove that lots of components of Ψ'_{ijk} vanish.
- Then, via a null rotation of n, we can move to a frame where $\Psi'_{ijk} = 0$.
- In this frame, can show that $\Omega'_{ij} = 0$, and hence the spacetime is Type D. (the new n is also a multiple WAND).

□

Sketch proof of geodesity theorem (3)

We can then go further with restricting the class of spacetimes that admit non-geodesic multiple WANDs:

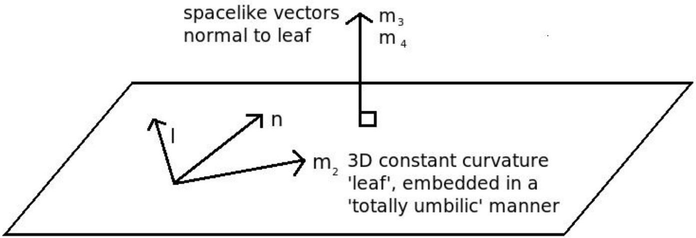
Theorem (*MND + Reall 2009*)

An Einstein spacetime that admits a non-geodesic multiple WAND is foliated by totally umbilic, constant curvature, Lorentzian, submanifolds of dimension three or greater, and any null vector field tangent to the leaves of the foliation is a multiple WAND.

[A totally umbilic submanifold has extrinsic curvature proportional to its induced metric. This property implies that it is “totally null geodesic”, i.e. any null geodesic of the submanifold is a null geodesic of the full spacetime.]

Cartoon of submanifold theorem

At each point, we can do something like this:



(This is a 5D picture, in arbitrary dimension the surface might have dimension ≥ 3.)

Sketch proof of submanifold theorem

This submanifold structure emerges as follows:

- Start in a null frame $\{\ell, n, m_i\}$ where ℓ is a non-geodesic multiple WAND, and apply a null rotation about n to get a new frame $\{\hat{\ell} = \ell - z_i m_i - \frac{1}{2}z^2 n, n, \hat{m}_i = m_i + z_i n\}$.
- Look to choose z_i such that $\hat{\ell}$ is also a multiple WAND, this requires
$$\hat{\Psi}_{ijk} \equiv \Phi_{iljk} z_l - 2\Phi_{i[j} z_{k]} = 0.$$
- This is a linear map acting on z_i, let K be the kernel of dimension N. By a Bianchi identity, know that $\kappa_i \in K$ is a solution, so $N \geq 1$.
- When $\hat{\Psi}_{ijk} = 0$, it turns out that $\hat{\Omega}_{ij} = 0$ too, so any non-geodesic multiple WAND is part of a continuous family of multiple WANDs $\hat{\ell}$.

Sketch proof of structure theorem (2)

Pick our spatial basis such that $m_2, \ldots, m_{N+1}$ is a basis for K, and consider the distribution spanned by
$e_A = \{\ell, n, m_2, \ldots, m_{N+1}\}$.

Properties of e_A

- $[e_A, e_B] \in \mathrm{span}(e_A)$, so the distribution is *integrable*, and hence defines submanifolds.
- These submanifolds have curvature

$$\tilde{R}_{ABCD} = 2\mathcal{R}\eta_{A[C}\eta_{D]B}$$

  for some known function $\mathcal{R}$.
- The $(N+2)$-dimensional Bianchi identity implies that $\mathcal{R}$ is constant on a submanifold.
- Extrinsic curvature is proportional to the metric η_{AB}.

Sketch proof of geodesity theorem (4)

We’re now basically done with the main theorem:

- Since these submanifolds are constant curvature they have vanishing Weyl tensor.
- Any null geodesic vector field lying tangent to a submanifold can be extended to the whole spacetime; certainly there are two independent vector fields with this property.
- *Any* such vector fields are multiple WANDs for the spacetime (intuitively think of them as being restricted to conformally flat directions).
- Hence, the spacetime admits a pair of geodesic multiple WANDs.

In 5 dimensions...

Theorem

A five-dimensional Einstein spacetime admits a non-geodesic multiple WAND if, and only if, it is locally isometric to one of the following:

1. *Minkowski, de Sitter, or anti-de Sitter spacetime*
2. *A direct product* $dS_3 \times S^2$ *or* $AdS_3 \times H^2$
3. *A spacetime with metric*

$$ds^2 = r^2 d\tilde{s}_3^2 + \frac{dr^2}{U(r)} + U(r)dz^2, \qquad U(r) = k - \frac{m}{r^2} - \frac{\Lambda}{4}r^2,$$

where $m \neq 0$, $k \in \{1, 0, -1\}$, $d\tilde{s}_3^2$ *is the metric of a 3d Lorentzian space of constant curvature with Ricci scalar* $6k$, *and the coordinate* r *takes values such that* $U(r) > 0$.

Summary

It exists...

These methods are out there, and not well-explored in higher dimensions.

Application to perturbations

Used successfully to find gauge invariant quantities for use in perturbation theory.

Future application?

Can we find new solutions using it?

Main Theorem

We can always find geodesic multiple WANDs for any algebraically special spacetime.

Main Theorem Missing

A complete generalization of Goldberg-Sachs.

Any questions?