Others applications to the Corvino-Schoen method

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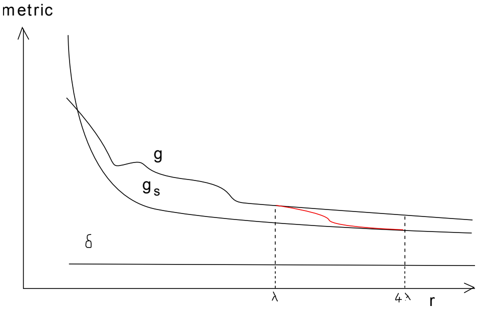
- Gluing of an AF metric g, $R(g) = 0$ with a Schwarzschild (slice) metric g_S on an annulus $B(4\lambda)\backslash B(\lambda)$, to a $R = 0$ metric.



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- Note : the linearized scalar curvature operator is an under determined elliptic operator.

$$Lh := DR(g)h = \text{div div } h + \Delta \text{ Tr } h - \langle \text{Ric}(g), h\rangle$$

$$L^* u = \text{Hess } u + \Delta u \; g - u \, \text{Ric}(g)$$

Remark : The kernel of L^* on $(\mathbb{R}^3, \delta)$ consists of affine functions so is 4 dimensional.

Here some examples

- div $: T^{p+1} \longrightarrow T^p$. Kernel : divergence free tensors. On $\mathbb{R}^3$, with $p = 0$ a model in the kernel is a point charge field :

$$E_Q = \frac{1}{4\pi} \frac{Q}{r^2} \frac{\vec{r}}{r} , \quad \vec{r} = (x, y, z) , \quad r = |\vec{r}|.$$

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- div^m $: \mathrm{T}^{p+m} \longrightarrow T^p$.

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- Look for U of the form

$$U = \zeta P^* u,$$

with u allowed to blow up a the boundary, but ζ vanishes more !

Isomorphism theorem

Let P be an under determined elliptic operator of order m.
Define the weighted spaces on Ω :

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- Then

$$\Pi_{K^\perp}\ e^{2/x}\ P\ x^{4m} e^{-2/x}\ P^* : K^\perp \cap H^{k+2m} \longmapsto H^k \cap K^\perp$$

is an isomorphism.

Idea of the Proof

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- Integrations by part shows that projection onto the kernel is trivial using (KRC).

Example of gluing

Let $\Omega_1 \subset \Omega_2 \subset \Omega_3$ with $\Omega = \Omega_2 \backslash \overline{\Omega}_1$ relatively compact. Let V and W (defined in Ω_3) in the kernel of P. One wants to glue V and W on Ω. Let χ equal 1 near $\partial\Omega_1$ and 0 near $\partial\Omega_2$. Let

$$T := \chi V + (1 - \chi) W$$

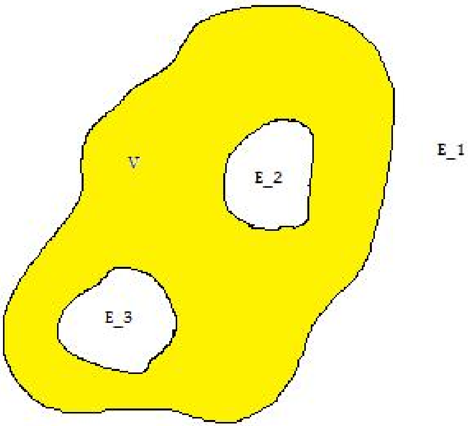
We then solve

$$PU = -PT =: f.$$

The glued solution is $T + U$. The necessary condition that f is orthogonal to K corresponds to the fact that V and W induce the same "flux" on, say, $\partial\Omega_2$.

Note : If the flux is zero, one can take V or W to be zero (this can be used for quotients or connected sum).

Gluing with some models on $\mathbb{R}^n$



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- In fact, whatever $R(g_1)$ and $R(g_2)$ are, one can also solve
$$R[\chi g_1 + (1-\chi)g_2 + h] = \chi R(g_1) + (1-\chi)R(g_2).$$
This gluing generalizes the Corvino type one, and keeps inequalities satisfied by the scalar curvature if any.