Superradiance, trapping, and decay for waves on Kerr spacetimes in the general subextremal case $|a| < M$

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joint work with I. Rodnianski

Recall the Kerr family $0 \le |a| \le M$

$$g_{M,a} = -\frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left(a\,dt - (r^2+a^2)d\phi\right)^2$$

$\rho^2 = r^2 + a^2\cos^2\theta,$

$\Delta = r^2 - 2Mr + a^2 = (r - r_-)(r - r_+),\ r_+ \ge r_-$

Vacuum solution: $\boxed{R_{\mu\nu} = 0}$ Killing fields $\boxed{\partial_t,\ \partial_\phi}$

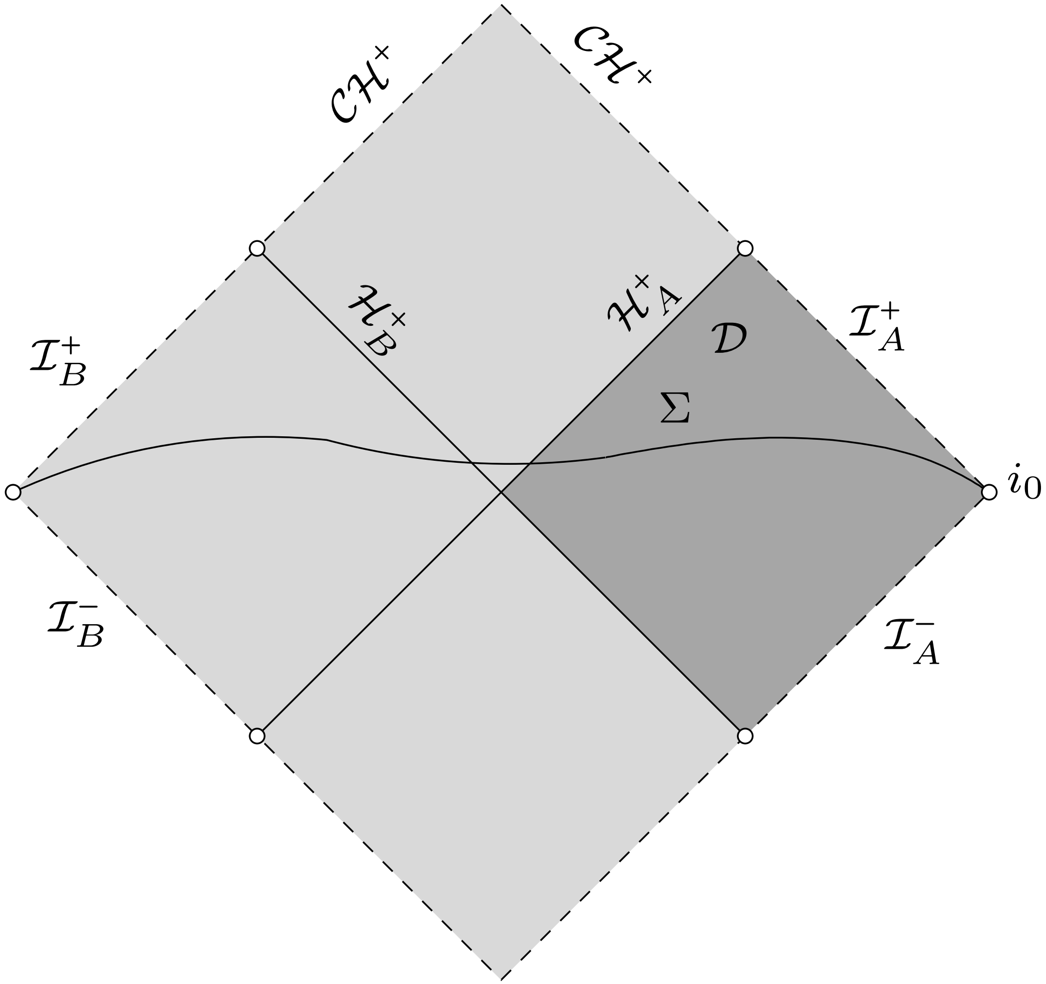
$a = 0$ Schwarzschild, 1916. General $a \neq 0$ case KERR, 1963

(t, r, θ, ϕ) Boyer-Lindquist coordinates.

Domain of outer communications: $r > r_+$.

$a = M$ (extremal)

Penrose diagramme for Kerr $(0 < a < M)$



Boundedness and decay for $\Box_g \psi = 0$ on Schwarzschild and Kerr:

Contributors include: Alinhac, Andersson–Blue, Bachelot–Bachelot, Blue–Soffer, Blue–Sterbenz, Carter, M. D.–Rodnianski, Donninger–Schlag–Soffer, Finster–Kamran–Smoller–Yau, Gundlach, Hartle–Wilkins, Kay–Wald, Kronthaler, Machedon–Stalker, Marzuola–Metcalfe–Tataru–Tohaneanu, Mason–Nicolas, Press–Teukolsky, Price, Pullin, Regge–Wheeler, Sa Barreto–Zworski, Tataru–Tohaneanu, Tataru, Twainy, Wald, Whiting

[See M. D.–Rodnianski *Lectures on black holes and linear waves*, arXiv:0811.0354]

Current state of the art for the 'quantitative' study of $\Box_g\psi = 0$

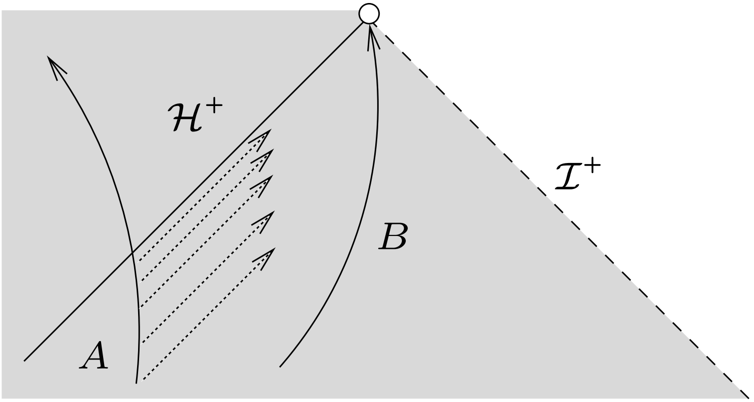
1. Boundedness in a general class of C^1 stationary axisymmetric spacetimes (M.D.–Rodnianski)
2. "Integrated local energy decay" for exactly Kerr:
   1. $|a| \ll M$ (M.D.–Rodnianski, Tataru–Tohaneanu, Andersson–Blue), and
   2. $\boxed{|a| < M \text{ (M.D.–Rodnianski) } \textbf{this talk}}$
3. Pointwise-in-time decay from 1. and 2. (energy-based method M.D.–Rodnianski, method based on the resolvent Tataru)

Review of the main features of Kerr spacetimes

1. Red-shift
2. Superradiance
3. Trapping

The red-shift

The redshift is classically understood in the geometric optics approximation in terms of signals sent by two observers A and B.



First discussed in the Schwarzschild setting by Oppenheimer–Snyder, 1939.

In fact, properly thought of, only depends on positivity of surface gravity.

Extremal case $a = M$: The red-shift factor at the horizon vanishes.

Superradiance

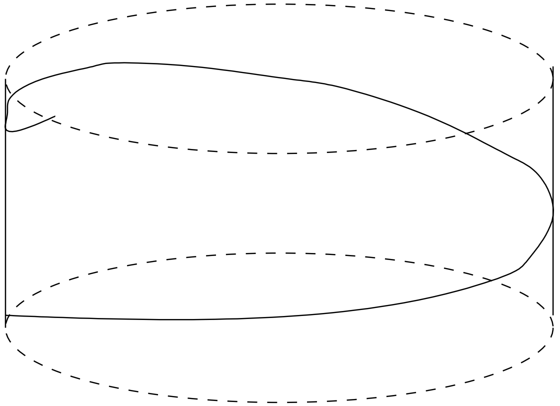
In Schwarzschild ($a = 0$), the Killing vector field ∂_t is timelike in the exterior, becoming null on the horizon. Thus there is a **conserved** *(by Noether)* **non-negative definite** *(by the timelike condition)* energy. The only subtlety is that this energy degenerates at the horizon.

In stationary perturbations of Schwarzschild, ∂_t in general becomes **spacelike** near the horizon. This happens already for Kerr with $0 \neq |a| \ll M$. The corresponding energy is conserved but does not have a sign. For particle motion, this leads to the so-called Penrose process. For waves, this leads to the phenomenon of *superradiance* (ZELDOVICH).

In particular, using the conservation law associated to ∂_t one cannot prove *a priori* boundedness, even away from the horizon.

Trapping

On Schwarzschild, the "photon sphere" $r = 3M$ has the property that it contains null geodesics. These null geodesics thus neither escape to $\mathcal{I}^+$ nor to the horizon $\mathcal{H}^+$.



In Kerr, the behaviour persists, but it is more complicated!

One can concentrate energy for arbitrarily large times near trapped null geodesics. One has to capture this to prove dispersive results.

In particular, pointwise-in-time decay estimates for energy must lose derivatives (Ralston).

Proof of integrated local energy decay

Will only discuss here the first energy. (Higher order estimates require commutation with the 'redshift' vector field, the Hawking vector field and ∂_t.)

The method of proof will exploit $\boxed{\text{energy currents.}}$

As we shall see, in the large a case, the construction of these currents will need to be 'frequency localised' for two reasons. (i) To distinguish between 'non-superradiant' and 'superradiant' frequencies, and (ii) to degenerate at the correct value of r.

A convenient way of doing both at the same time is frequency localising via Carter's celebrated $\boxed{\text{separation of the wave equation.}}$

Note that in view of Ricci flatness, this separability is equivalent to separability of Hamilton-Jacobi equations and the existence of a Killing tensor.

Separation: Let $\Box_g \Psi = F$,

$$\widehat{\Psi}(\omega, r, \theta, \phi) = \int_{-\infty}^{\infty} \Psi(t, r, \theta, \phi)\, e^{-i\omega t} dt,$$

$$P(a\omega)\, S_{m\ell}(a\omega, \cos\theta) e^{im\phi} = \lambda_{m\ell}^{(a\omega)}\, S_{m\ell}(a\omega, \cos\theta) e^{im\phi},$$

where

$$P(\xi)\, f = -\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} f \right) - \frac{\partial^2 f}{\partial\phi^2} \frac{1}{\sin^2\theta} - \xi^2 \cos^2\theta f.$$

Then

$$\widehat{\Psi}(\omega, r, \theta, \phi) = \sum_{m\ell} \Psi_{m\ell}^{(a\omega)}(r) S_{m\ell}(a\omega, \cos\theta) e^{im\phi}.$$

$$\Delta \frac{d}{dr} \left(\Delta \frac{\Psi_{m\ell}^{(a\omega)}}{dr} \right) + \left(a^2 m^2 + (r^2 + a^2)^2 \omega^2 - \Delta(\lambda_{m\ell}^{(a\omega)} + a^2 \omega^2) \right) \Psi_{m\ell}^{(a\omega)}$$
$$= (r^2 + a^2) \Delta\, F_{m\ell}^{(a\omega)}. \tag{1}$$

$$u_{m\ell}^{(a\omega)}(r) = (r^2 + a^2)^{1/2}\Psi_{m\ell}^{(a\omega)}(r), \qquad H_{m\ell}^{(a\omega)}(r) = \frac{\Delta F_{m\ell}^{(a\omega)}(r)}{(r^2 + a^2)^{1/2}}.$$

$$\frac{dr^*}{dr} = \frac{r^2 + a^2}{\Delta}, \qquad {}' = \frac{d}{dr^*}$$

$$(u_{m\ell}^{(a\omega)})'' + (\omega^2 - V_{m\ell}^{(a\omega)}(r))u = H_{m\ell}^{(a\omega)}$$

$$\begin{aligned} V_{m\ell}^{(a\omega)}(r) \quad &= \quad \frac{4Mram\omega - a^2m^2 + \Delta(\lambda_{m\ell}^{(a\omega)} + \omega^2 a^2)}{(r^2 + a^2)^2} \\ &\quad + \frac{\Delta(3r^2 - 4Mr + a^2)}{(r^2 + a^2)^3} - \frac{3\Delta^2 r^2}{(r^2 + a^2)^4}. \end{aligned}$$

Superradiant frequencies: $0 \le \omega m < \frac{am^2}{2Mr_+}$.

Completely separated energy current identities (analogues of $\nabla^{\mu}(\mathbf{T}_{\mu\nu}[\psi]y\partial_{r^*}{}^{\nu}) = T_{\mu\nu}[\psi]^{y\partial_{r^*}}\pi^{\mu\nu}$, etc.)

$$\text{ϟ}^{y}[u] = y\left(|u'|^2 + (\omega^2 - V)|u|^2\right),$$

$$(\text{ϟ}^{y}[u])' = \underline{y'\left(|u'|^2 + (\omega^2 - V)|u|^2\right) - yV'|u|^2} + 2y\,\mathrm{Re}(u'\bar{H}),$$

$$\text{ϙ}^{h}[u] = h\mathrm{Re}(u'\bar{u}) - \frac{1}{2}h'|u|^2.$$

$$(\text{ϙ}^{h}[u])' = \underline{h\left(|u'|^2 + (V - \omega^2)|u|^2\right) - \frac{1}{2}h''|u|^2} + h\,\mathrm{Re}(u\bar{H}).$$

$$\mathrm{Q}^{f}[u] = \text{ϙ}^{f'}[u] + \text{ϟ}^{f}[u] = f\left(|u'|^2 + (\omega^2 - V)|u|^2\right) + f'\mathrm{Re}\,(u'\bar{u}) - \frac{1}{2}f''|u|^2.$$

$$(\mathrm{Q}^{f}[u])' = \underline{2f'|u'|^2 - fV'|u|^2 - \frac{1}{2}f'''|u|^2} + \mathrm{Re}(2f\bar{H}u' + f'\bar{H}u).$$

$$\text{Ͳ}^{d}[u] = d|u' - i(\omega - (am/2Mr_+)u|^2 + d(\omega^2 - V - |\omega - (am/2Mr_+)|)|u|^2$$

$$(\text{Ͳ}^{d}[u])' = \ldots$$

General idea:

From the above currents, produce integral identities with positive definite underlined bulk terms and (upon summation) controlled boundary terms.

In the Schwarzschild case, this was first achieved by various authors using just spherical harmonics, and later, without any frequency decomposition in M.D.–RODNIANSKI 2007. **All these constructions involve fine-tuning of parameters to handle lower order terms, and give the impression of 'barely' working.**

With the complete separation, it turns out that suitable currents can be constructed systematically, *without fine-tuning parameters* involved in defining template functions f, h, y, etc. Only actual physical obstructions appear as difficulties, and these are reflected cleanly in the construction of f, h, y.

Schwarzschild

If $\omega = 0$ (formally) apply $\text{Ϙ}^h[u]$ with $h = 1$, noting $V \geq 0$. Can perturb this construction using the help of the redshift to yield positive definite bulk for $\omega^2 < b_1$, for sufficiently small b_1.

If $b_1 \leq \omega^2 < \infty$, $\lambda_{m\ell} = \ell(\ell+1) \leq B_2$, B_2 arbitrary, then apply $\text{ϟ}^y[u]$ with $y = \exp \int \lambda V dr^*$ for a large parameter λ depending on B_2. Boundary terms controlled by the conserved energy.

For large $\lambda_{m\ell} \ll \omega^2$, there is again an easy construction based on $\text{ϟ}^y[u]$ for suitable y.

In general, for large $\lambda_{m\ell}$, note that V' has a unique simple 0.

$\boxed{\text{Choose } \mathrm{Q}^f[u] \text{ with } f \text{ vanishing at the zero of } f.}$

IMPORTANT: In all cases, boundary terms will be controlled by the conserved energy upon summation.

Kerr for small a

For large ω^2, the statement that V' has a unique simple zero is stable in the frequency range $1 \ll \omega^2 \sim \lambda_{m\ell}$.

The constructions for all the other frequency ranges can be easily perturbed to yield positive definite bulks.

The boundary terms, however, are not a priori controlled–this is the problem of superradiance.

Since $|a|$ is small, this can be remedied as follows: add a small amount of the redshift identity corresponding to $\overset{\lrcorner\,\llcorner}{\top}{}^{d}[u]$ to correct for the bad boundary term, absorbing the resulting small error term in the positive definite bulk.

In this argument, the requirement of smallness of a is fundamental. The case of large a requires a new physical insight.

Kerr for general a

It is only for large a where one must really come to terms with the problem of superradiance.

Assume for the time being that m is sufficiently large. Note then that $\lambda_{m\ell} + \omega^2 a^2 \geq m(m+1)$ is also large.

Then

$$V \sim \frac{4Mram\omega - a^2 m^2 + \Delta(\lambda_{m\ell} + \omega^2 a^2)}{(r^2 + a^2)^2}$$

Proposition 1. *In the superradiant regime* $0 \leq m\omega \leq \frac{am^2}{2Mr_+}$, V *has a unique critical point* $r_{\max}$*, which is a global maximum, and* $V(r_{\max}) > \omega^2$

This allows us to construct a current (giving rise to a positive definite bulk) of the form

$$\mathrm{Q}^f[u] + C_1 \text{⫧}^d[u] + C_2 \text{♀}^h[u]$$

We construct $\mathrm{Q}^f[u]$ with f as usual vanishing linearly at $r_{\max}$. We then add as much as $\text{⫧}^d[u]$ is necessary so as to make the boundary term on the horizon of a good sign, where d is cutoff in the region where $V(r) > \omega^2$. We then add a large multiple of a $\text{♀}^h[u]$ where h is entirely supported in $\{V(r) > \omega^2\}$ to absorb the first order terms of the wrong sign that enter via the cutoff on d.

This yields a current yielding positive definite bulk terms (w/o degeneration) and positive definite boundary terms.

In particular, the above proposition says that 'superradiant frequencies' are not trapped.

Remark 1. In the case $|a| \ll M$, this fact was used in our *boundedness* paper, where it could be inferred however, from stability considerations from Schwarzschild.

Remark 2. The above statement is interesting even when restricted to $\omega = 0$. It reflects the fact (at the level of geometric optics) that zero-energy null geodesics are not trapped. In fact, such geodesics all travel from $\mathcal{H}^-$ to $\mathcal{H}^+$. It is pricisely this phenomenon which is implicitly used by IONESCU–KLAINERMAN to prove unique continuation for stationary solutions to the wave equation on Kerr. Further applications by ALEXAKIS–IONESCU–KLAINERMAN to 'uniqueness of Kerr'.

Proposition 2. *In the non-superradiant regime* $m\omega < 0$ *or* $\omega \geq \frac{a|m|}{2Mr_+}$*, then* V *in general has at most two critical points* $r_{\min} < r_{\max}$*. Moreover* $V(r_{\min}) \leq \omega^2$*.*

Using the above, one can construct a current to obtain a positive definite bulk term as follows.

As usual, we consider $\mathrm{Q}^f[u]$ where f changes sign at $r_{\max}$. We now add the identity of the current $\lightning^y[u]$, applied only in $[r_+, r_{\min}]$, with a $y = -1$ at $r_{\min}$, suitably constructed to absorb the bad term where fV' term in $(r_+, r_{\min})$. Note that the boundary term at $r = r_{\min}$ has a favourable sign.

We need not worry about the remaining boundary terms as they will be controlled by the conserved energy which is positive definite when restricted to non-superradiant frequencies.

The assumption m sufficiently large can be replaced by the assumption $\lambda_{m\ell}$ sufficiently large. Also, the case where ω dominates can be easily handled as in Schwarzschild.

This leaves a compact frequency range.

For non-superradiant low frequencies, we can use the Schwarzschild argument for such frequencies.

Superradiant low frequencies need a separate argument: This can be treated for instance by a slight refinement of a classical result of WHITING on mode stability.

Various technical issues:

When can you take the Fourier transform?

What about terms arising from the cutoff?

For this, convenient to apply a continuity argument in a and techniques introduced in our original boundedness paper.

Some other important results

$\Lambda > 0$ (Bony–Häfner, M.D.–Rodnianski, Melrose–Sa Barreto–Vasy for Schw–de Sitter, Dyatlov for Kerr–de Sitter)

$\Lambda < 0$ (Holzegel for Kerr–AdS and perturbations, Baskin in progress)

Mode stability, quasinormal modes on Schwarzschild, Kerr, Schwarzschild-de Sitter, etc. (Whiting, Bachelot, Zworski–Sa Barreto, Dyatlov, . . .)

Dirac, Maxwell equations (Blue, Häfner, Finster et al)

Tails for fixed spherical harmonics on Schwarzschild (Price, Gundlach, M.D.–Rodnianski, Machedon–Stalker, Bizon et al, Kronthaler, Donninger–Schlag–Soffer. . .)

Open problems and future directions

1. The extremal case $a = M$ (recent results of S. ARETAKIS)
2. Higher dimensions (recent results of SCHLUE)
3. Other measures of decay, Strichartz estimates, and applications to power-law nonlinearities etc. (TOHANEANU, BASKIN, . . .)
4. Robust additional decay with applications to non-linear problems (LUK)
5. Maxwell equations on Kerr (BLUE, . . .)
6. The equations of gravitational perturbation (previous talk of HOLZEGEL)

and of course

Nonlinear stability of Kerr...