Tiling Cohomology

and Quasiperiodic Baked Goods

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Three key questions

For every mathematical concept:

- What is it?

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- How do you compute it?

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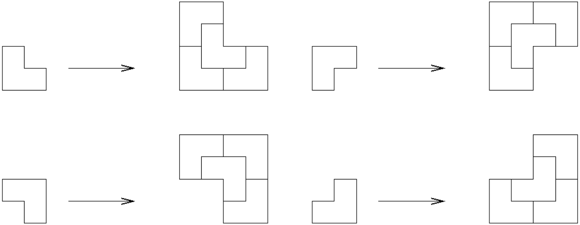
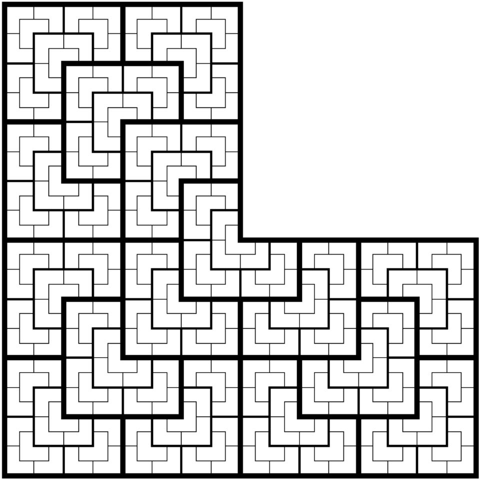
For every mathematical concept:

- What is it?
- How do you compute it?
- Why in blazes should you care?

Puzzle 1: Mass transport



Musical chairs



Three different mass distributions

- f_1 puts 2 kg on every tile that sits in the standard L configuration, i.e. missing the northeast corner, and no mass on the other three kinds of tiles.
- f_2 puts 1 kg on every tile that is missing a NE or SW corner, and none on tiles that are missing NW or SE corners.
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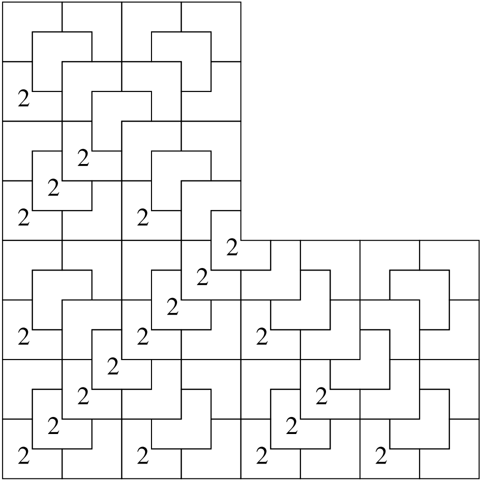
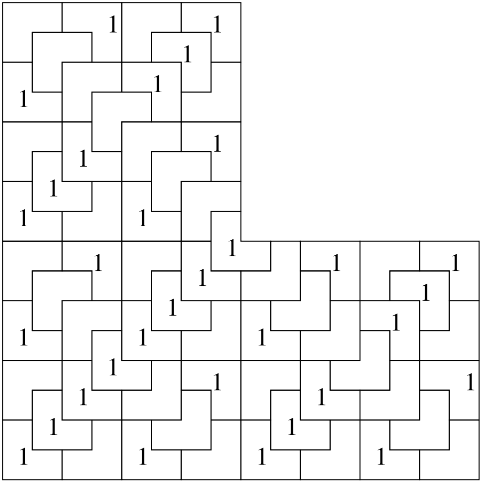
- All three distributions have overall density 0.5 kg/tile. Which are related by bounded/wPE/sPE transport?

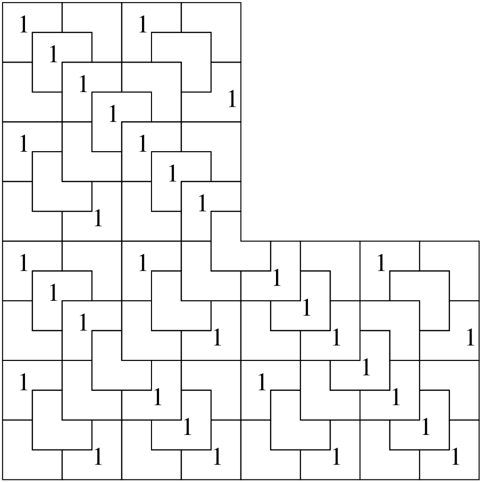
2 kg on the NE chairs



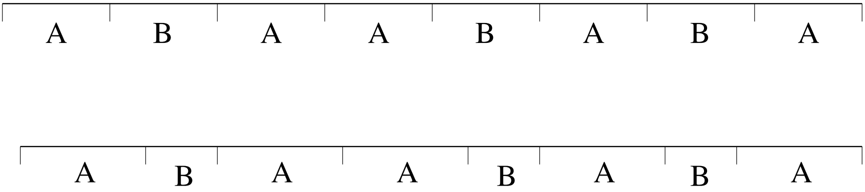
1 kg on the NE and SW chairs



1 kg on the NW and SE chairs

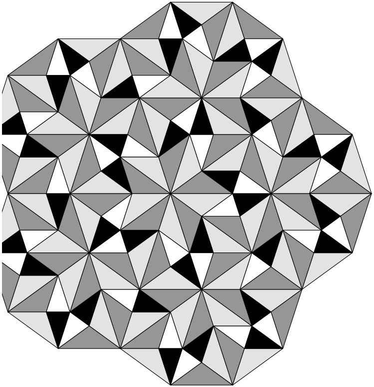


Puzzle 2: Fibonacci shape changes



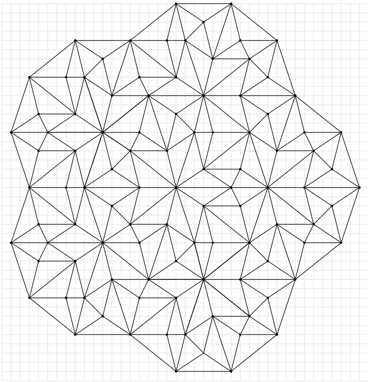
How are these tilings related? How do their diffraction patterns compare?

Puzzle 3: Penrose shape changes

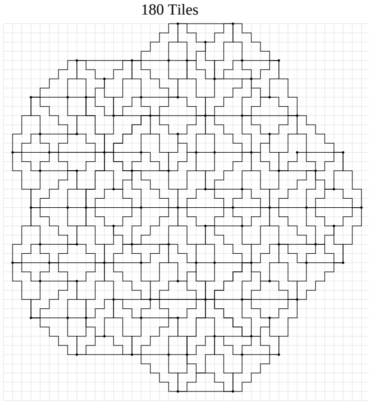


Rational Penrose

180 Tiles



Squared off Penrose



Puzzle 4: Ergodic averages

Thue-Morse tiling: $A \to AB$, $B \to BA$,

$\ldots ABBABAABBAABABBAABBABAABBAABABBABAABABBA \ldots$

What are the maximum/minimum number of times that the pattern ABA appears in a sub-word of length N? How does the variation scale with N?

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FLC tiling metric

- Idea for FLC tilings: Two tilings with the same set of tile types are ϵ close if they agree on $B_{1/\epsilon}$, up to an ϵ translation.

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- (We won't go there)

Continuous Hulls

Simplest way to build a tiling space:

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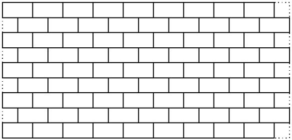
- Start with an FLC tiling T.
- Consider the set $\{T - x\}$ of translates of T.
- $\Omega_T = \overline{\{T - x\}}$. $T' \in \Omega_T$ iff every patch of T' appears somewhere in T.
- Orbit closure of T = Tiling space of T = Continuous hull of T.

Hulls of periodic tilings



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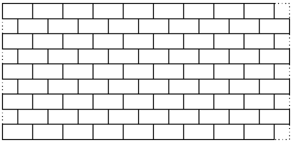

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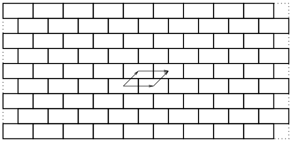

A torus!

A non-periodic example

$$T = \dots AAAA.BBBB \dots \text{“=”} \; A^\infty . B^\infty .$$

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- As $x \to \infty$, $T - x$ approaches periodic $\dots BBBBB \dots$ tiling. Limiting circle.
- Hull $=$ slinky! Connected but not path-connected.

Local topology

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- Restrict T to $B_{1/\epsilon}$.
- Move T by up to ϵ: continuous degrees of freedom.
- Fill out near ∞. Discrete choices.
- Neighborhood $\sim B_\epsilon \times C$.

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Inverse limits in general

If $X_0, X_1, \ldots$ are spaces and $\rho_n : X_n \to X_{n-1}$ are continuous maps,

$$X = \varprojlim X_i := \{(x_0, x_1, \ldots) \in \prod X_n | \rho_n(x_n) = x_{n-1} \forall n\}.$$

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X has the product topology. $(x_0, x_1, \ldots)$ is close to $(y_0, y_1, \ldots)$ if $x_i \approx y_i$ for all $i \leq N$. I.e. if $x_N \approx y_N$.

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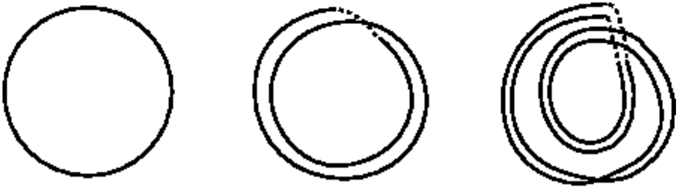

Γ_0 Γ_1 Γ_2

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- Many different schemes: different details, (mostly) same strategy.
- $\varprojlim \Gamma_n =$ consistent instructions for tiling bigger and bigger regions, i.e. instructions for a complete tiling.
- So how do instructions for partial tilings turn into a CW complex?!

Anderson-Putnam Complex

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Collared tiles

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- Can be repeated to get n-times collared tiles.

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- Collared tiles have same size as regular tiles, but carry more info.

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- $\Omega = \varprojlim \Gamma^n$.
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- Calculationally not so much, since Γ^n's are all different.

Substitution tilings

1-dimensional example (Fibonacci) : $a \rightarrow ab$, $b \rightarrow a$.

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A word is legal if it sits inside one of these patterns.
A bi-infinite word is legal if every sub-word is legal.
Make into self-similar tilings by assigning length $(1+\sqrt{5})/2$ to a tile and 1 to b tile.

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- To get border forcing, collar once (if necessary).

Other techniques

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- (Forest-Hunton-Kellendonk have a different sort of inverse limit construction for cut-and-project tilings)
- Can express tilings with infinite local complexity as inverse limits, too. Details depend on setting.

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Pattern-equivariant functions and forms

- Given a tiling T, a function $f(x)$ on $\mathbb{R}^n$ is *strongly pattern-equivariant* (sPE) if $\exists R > 0$ s.t. x depends only on tiling on $B_R(x)$. (Think: finite range potentials)
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- Strongly/weakly PE forms are strongly/weakly PE functions times $dx^i \wedge dx^j \wedge \cdots$.
- If α is a PE form, so is $d\alpha$.
- $H^k_{PE}(T)$ = closed sPE k-forms / d(sPE $k-1$ forms).

Pattern-equivariant cochains

- A tiling T gives a decomposition of $\mathbb{R}^n$ into vertices, edges, 2-cells, 3-cells, etc. Tiles are n-cells. Orient the cells arbitrarily.
- A (real-valued) k-cochain assigns a real number to each oriented k-cell. A mass distribution is just an n-cochain.
- k-cochains can be sPE or wPE.
- Coboundaries: If α is a k-cochain, and c is a $(k+1)$-cell, then $(\delta\alpha)(c) := \alpha(\partial c)$.
- If α is wPE/sPE, so is $\delta\alpha$.
- Let Ω^k_w and Ω^k_s denote the weakly and strongly PE k-cochains on T.

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- $H^k_{PE}(T) = \dfrac{\text{Closed } k\text{-cochains}}{\text{Exact } k\text{-cochains}}$ (Same answer as with forms!)
- A cohomology class is *asymptotically negligible (AN)* if it can be respresented by a weakly exact cochain/form.

A topological invariant

Theorem (Kellendonk-Putnam, S)

If T is a repetitive tiling, then H^k_{PE} is canonically isomorphic to the k-th real-valued Čech cohomology $\check{H}^k(\Omega_T)$, where Ω_T is the continuous hull of T. In particular, all tilings in Ω_T have the same PE cohomology.

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- Strategy: Write tiling space Ω as inverse limit of CW complexes Γ_i. Then

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- Complicated definition involving combinatorics of open covers.
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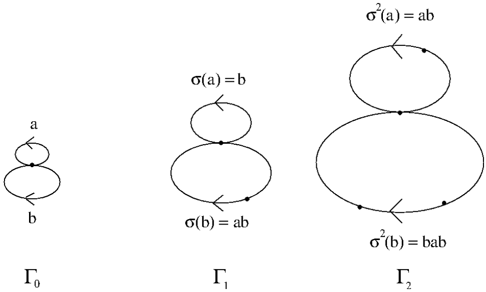
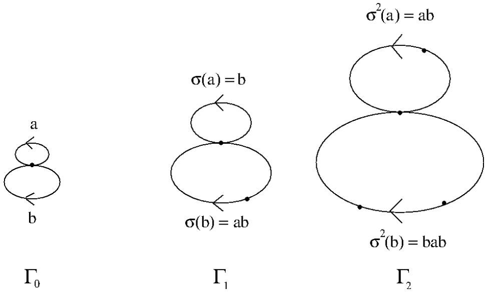
But we already did that!

Fibonacci


a
b
$\sigma(a) = b$
$\sigma(b) = ab$
$\sigma^2(a) = ab$
$\sigma^2(b) = bab$
Γ_0
Γ_1
Γ_2

Fibonacci



$$H^1(\Gamma_n) = \mathbb{Z}^2; \qquad H^1(\Omega) = \lim(\mathbb{Z}^2, \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)) = \mathbb{Z}^2.$$

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- Substitution acts as $\times 2$ on H^1, as matrix with eigenvalues $4, 2, 2$ on H^2.
- $H^1(\Omega) = \mathbb{Z}[1/2]^2, \qquad H^2(\Omega) = \frac{1}{3}\mathbb{Z}[1/4] \oplus \mathbb{Z}[1/2]^2.$

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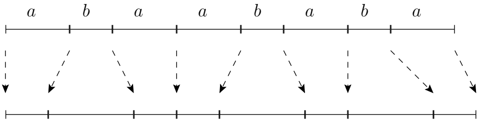
1D shape changes



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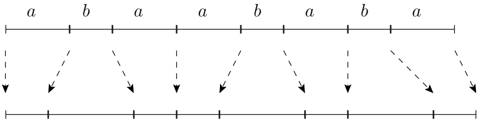

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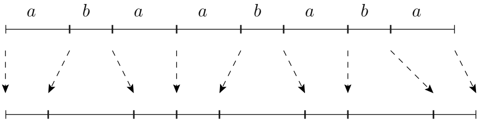

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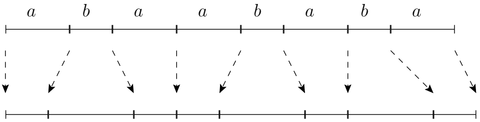
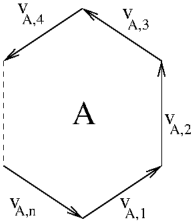


- Combinatorics of T_1 and T_2 are identical.
- Dynamics may be different.
- Some (but not all!) shape changes are topological conjugacies.

Shapes in 2 or more dimensions (Clark-S)



The shape of an *n*-gon is determined by the *n* vectors that describe the edges.

Parametrizing shape

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- But that's the same as a closed vector-valued 1-cochain on the Anderson-Putnam complex!

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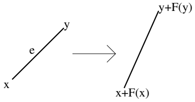
Modding out by MLD



- MLD equivalence moves each vertex x by $F(x)$, where $F : \mathbb{R}^n \to \mathbb{R}^n$ is an sPE function.

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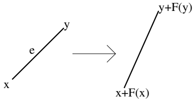

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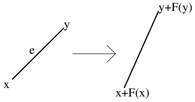

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$$\frac{\text{Shape changes}}{\text{MLD}} = \frac{\text{Closed sPE 1-cochains}}{\delta(\text{sPE 0-cochains})}$$

$$= H^1_{PE}(T, \mathbb{R}^n) = \check{H}^1(\Omega_T, \mathbb{R}^n).$$

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- Moving points by wPE amounts induces topological conjugacies, so H^1_{AN} describes shape changes that are topological conjugacies but not MLD.
- Theorem (Gottschalk-Hedlund, Kellendonk-S): A closed sPE 1-cochain is AN if and only if its integral is bounded.

Fibonacci is rigid

- Fibonacci tiling has $\phi = (1+\sqrt{5})/2$ "a" tiles for every "b" tile.
- If $\alpha(a) = 1$ and $\alpha(b) = -\phi$, α is AN.
- $H^1(\Omega_{Fib}, \mathbb{R}) = \mathbb{R}^2 = H^1_{AN} \oplus \mathbb{R}$.
- All shape changes for Fibonacci are a combination of topological conjugacy and overall rescaling.
- Dynamical properties of Fibonacci (e.g. pure point spectrum) unchanged by shape changes.

AN classes for substitutions

Setting: Ω is a substitution tiling space with a substitution map $\sigma : \Omega \to \Omega$.

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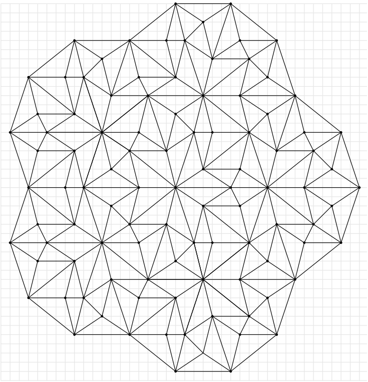
- All shape changes that preserve 180 degree rotational symmetry are combinations of rigid linear transformations and topological conjugacies, and preserve dynamics.

Rational Penrose

180 Tiles



AN classes for cut-and-project

Theorem (Kellendonk-S)

If T is a cut-and-project tiling of dimension n and codimension k, and if the "window" is a finite union of polyhedra, then $H^1_{AN}(\Omega_T, \mathbb{R}) = \mathbb{R}^k$.

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Theorem (Kellendonk-S)

Shape conjugacies of cut-and-project sets with polygonal windows are MLD to "reprojections". Same total space, lattice, same window, different projection to $\mathbb{R}^n$.

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- Deviations in count of *aba* are unbounded. (Actually grow as $\ln(N)$.)
- Nothing special about *aba*. Same thing applies to almost any pattern. (Just not *a* or *b*).

Cohomological answers to transport questions

If f_1 and f_2 are mass distributions on T, then f_1 and f_2 are closed and define cohomology classes $[f_1]$ and $[f_2]$. Then

- Theorem: There is a bounded transport from f_1 to f_2 if and only if $[f_1 - f_2]$ is well-balanced. (I.e. $\left\| \int_R f_1 - f_2 \right\| \leq c \|\partial R\|$.)

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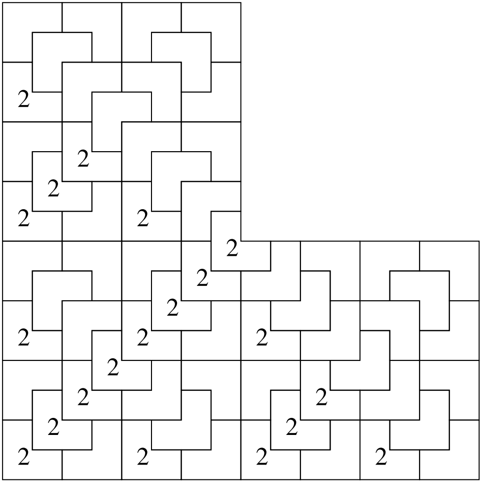
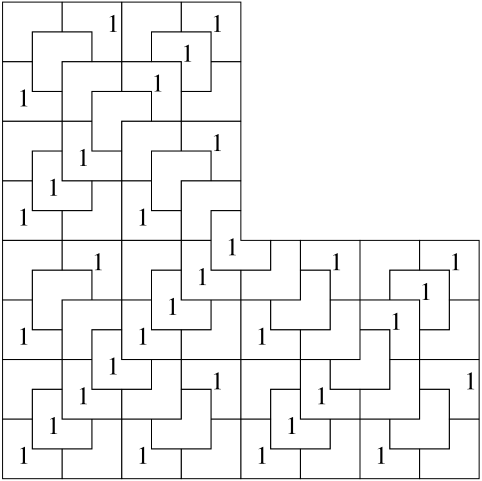
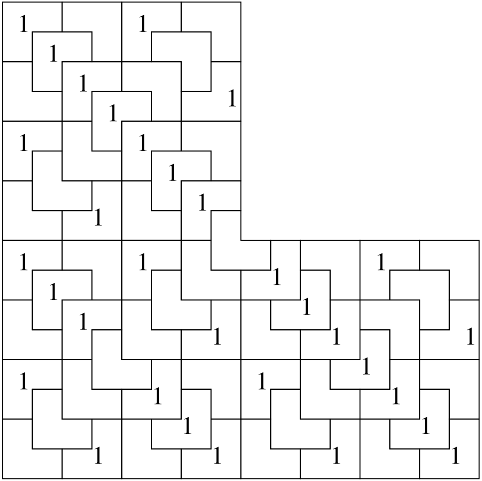
2 kg on the NE chairs



1 kg on the NE and SW chairs



1 kg on the NW and SE chairs



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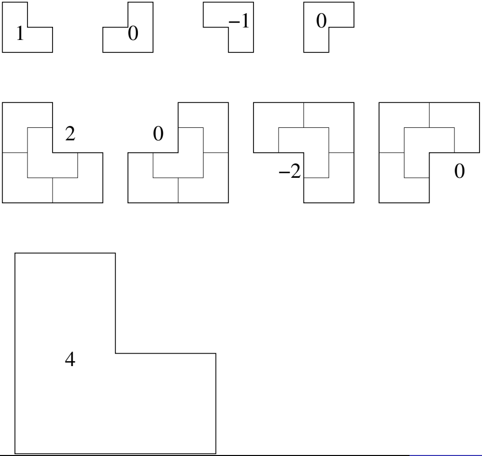
- (Last generator counts NW minus SE.)
- Remaining question: Is $f_1 - f_2$ well-balanced?

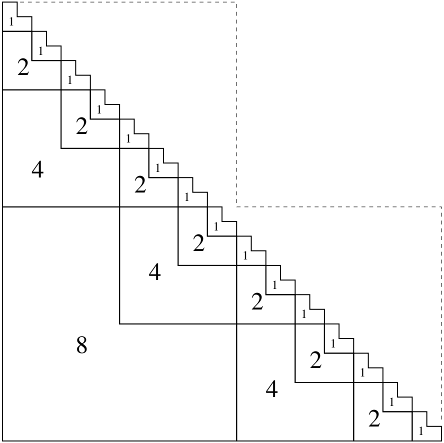
Scaling properties

Under substitution, $f_1 - f_2$ doubles at each stage:



N log N

On triangle of side length $N = 2^m$, $f_1 - f_2$ goes as $m2^m$.



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- Lots of other applications of cohomology, but we're out of time (and sliced bread).

Thank You!