Relating the Vogan and Satake forms in the quantum setting

Marco Matassa (joint work with Kenny De Commer)

Vrije Universiteit Brussel (VUB)

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Definition

A Lie $*$-algebra is a complex Lie algebra $\mathfrak{g}$ with an anti-linear, anti-multiplicative involution $* : \mathfrak{g} \to \mathfrak{g}$.

Correspondence Lie $*$-algebras $\Longleftrightarrow$ real Lie algebras, given by

$$(\mathfrak{g}, *) \mapsto \mathfrak{g}_* = \{x \in \mathfrak{g} : x^* = -x\},$$
$$\mathfrak{s} \mapsto (\mathfrak{s}_{\mathbb{C}}, *), \quad (x + iy)^* = -x + iy, \quad x, y \in \mathfrak{s}.$$

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In the following we will fix a compact real form of $\mathfrak{g}$, denoted by $\mathfrak{u}$, and write $*$ for its corresponding $*$-structure.

Let σ be a Lie algebra involution of $\mathfrak{u}$. Then $x^\dagger = \sigma(x)^*$ defines a $*$-structure. All real semisimple Lie algebras arise as $\mathfrak{g}_\dagger$.

We recover the usual correspondence real forms of $\mathfrak{g} \Longleftrightarrow$ involutions of $\mathfrak{u}$.

There are two standard forms for an involution:

1. Vogan form (from max. compact Cartan),
2. Satake form (from max. non-compact Cartan).

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Definition

Vogan form $\nu = \nu(Y, \mu)$. Data $Y \subseteq I$ and $\mu \in \mathrm{Out}(\mathfrak{g})$ and

$$\nu(h_i) = h_{\mu(i)}, \quad \nu(e_i) = \epsilon_i e_{\mu(i)}, \quad \nu(f_i) = \epsilon_i f_{\mu(i)},$$

Here $\epsilon : I \to \{\pm 1\}$ and $\epsilon_i = -1$ for $i \in Y$.

We will not recall all the details of the Satake form.

Definition

Satake form $\theta = \theta(X, \tau, z)$: $X \subseteq I$, $\tau \in \mathrm{Out}(\mathfrak{g})$ and z a unitary character on Q, satisfying various conditions. Then

$$\theta = \mathrm{Ad}(z) \circ \tau \circ \omega \circ \mathrm{Ad}(m_X).$$

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Example

Consider the non-trivial involution of $\mathfrak{sl}_2$. Then

$$\nu(h) = h, \quad \nu(e) = -e, \quad \nu(f) = -f,$$
$$\theta(h) = -h, \quad \theta(e) = -f, \quad \theta(f) = -e.$$

In this case $\mu = \tau = \mathrm{id}$, $Y = \{1\}$ and $X = \emptyset$.

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Also write U for the Lie group integrating $\mathfrak{u}$. We have

$$U = \{g \in G : g^* = g^{-1}\}.$$

Then U can be recovered from the $*$-characters of $\mathcal{O}(G)$, where

$$f^*(g) = \overline{f((g^{-1})^*)}.$$

Note that this procedure uses the Lie $*$-algebra structure.

Now let θ be an involution in Satake form and ν an involution in Vogan form, which we assume to be inner conjugate.

Moreover this can be realized by unitary elements. That is we have

$$\theta = \mathrm{Ad}(v) \circ \nu \circ \mathrm{Ad}(v^*), \quad v \in U.$$

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Define the element

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Then for any $g \in G$ we have

$$\theta(g) = k\nu(g)k^*.$$

Moreover the element k satisfies

$$k^* = \nu(k).$$

Given an involution σ, define the σ-twisted Hermitian elements by

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Back to θ and ν. There is a U-equivariant isomorphism

$$(H_\theta(G), \mathrm{Ad}_\theta) \cong (H_\nu(G), \mathrm{Ad}_\nu), \quad x \mapsto xk.$$

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$$(H_\theta(G), \mathrm{Ad}_\theta) \cong (H_\nu(G), \mathrm{Ad}_\nu), \quad x \mapsto xk.$$

We also have a U-equivariant embedding

$$(U/U^\theta, L) \hookrightarrow (H_\theta(G), \mathrm{Ad}_\theta), \quad xU^\theta \mapsto x\theta(x)^*.$$

Then we can consider the composite map

$$(U/U^{\theta}, L) \hookrightarrow (H_{\nu}(G), \mathrm{Ad}_{\nu}),$$

which is given explicitely by

$$uU^{\theta} \mapsto u\theta(u)^{*}k = uk\nu(u)^{*} = \mathrm{Ad}_{\nu}(u)k.$$

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Dualizing we obtain a surjective map

$$(\mathcal{O}(H_\nu(G)), \mathrm{Ad}_\nu^*) \twoheadrightarrow (\mathcal{O}(U/U^\theta), L^*).$$

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Goal: quantize this map (in an equivariant way).

We will show that this is possible. On the other hand the algebras $\mathcal{O}(H_\theta(G))$ and $\mathcal{O}(U/U^\nu)$ do not admit natural quantizations.

We start from the quantized enveloping algebra $U_q(\mathfrak{g})$. We write

$$\mathcal{U} := (U_q(\mathfrak{g}), *).$$

We write $\mathcal{A} := \mathcal{U}^\circ$ for the dual Hopf algebra of matrix coefficients.

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We have a map $I : \mathcal{A} \to \mathcal{U}$ given by

$$I(\omega) = (\omega \otimes \mathrm{id})(\mathscr{R}^* \mathscr{R}).$$

We get $\mathcal{U}$-module isomorphism $\mathcal{A} \cong \mathcal{U}_{\mathrm{fin}}$. Not an algebra isomorphism.

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However we can modify the product of $\mathcal{A}$ and consider the braided version $\mathcal{A}^{\mathrm{br}}$. Then we have a $*$-algebra isomorphism $\mathcal{A}^{\mathrm{br}} \cong \mathcal{U}_{\mathrm{fin}}$.

We want a quantum analogue of $(\mathcal{O}(H_\nu(G)), \mathrm{Ad}^*_\nu)$. Observe that:

1. $\mathcal{A}$ is not a $\mathcal{U}$-module algebra for Ad^*, while $\mathcal{A}^{\mathrm{br}}$ is,
2. on the other hand we want Ad^*_ν, need a twist!

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2. on the other hand we want Ad^*_ν, need a twist!

We will use some ideas from the paper [DCNTY (18)].

We have a straightforward quantization of the Vogan form:

$$\nu_q(K_\omega) = K_{N(\omega)}, \quad \nu_q(E_i) = \epsilon_i E_{\mu(i)}, \quad \nu_q(F_i) = \epsilon_i F_{\mu(i)},$$

where N is dual to $\nu|_{\mathfrak{h}}$. It is a Hopf $*$-algebra involution.

We can use ν_q to twist the Drinfeld double $\mathcal{D}$.

Consider the usual skew-pairing $(\cdot,\cdot) : \mathcal{U}^- \otimes \mathcal{U}^+ \to \mathbb{C}$. We set

$$(X, Y)_+ := (X, Y), \quad (X, Y)_- := (\nu_q(X), Y).$$

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For $\mu, \mu' \in \{\pm\}$ we define an algebra $\mathcal{D}_{\mu,\mu'}$ as follows. It is $\mathcal{U}^+ \otimes \mathcal{U}^-$ as a vector space, contains $\mathcal{U}^\pm$ as subalgebras, and has cross-relations

$$kh = (h_{(1)}, k_{(1)})_\mu h_{(2)} k_{(2)} (S(h_{(3)}), k_{(3)})_{\mu'}, \quad h \in \mathcal{U}^-, \ k \in \mathcal{U}^+.$$

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They fit together into a Hopf-Galois system. We will write

$$\mathcal{D} = \mathcal{D}_{++}, \quad \mathcal{D}_\nu := \mathcal{D}_{+-}.$$

Upon identifying the two Cartan parts of $\mathcal{D}$ we get $\mathcal{U}$.

In a similar way we can twist $\mathcal{A}^{\mathrm{br}}$ to obtain $\mathcal{A}^{\nu-\mathrm{br}}$. It becomes a $\mathcal{U}$-module algebra with respect to the twisted adjoint action Ad^*_ν.

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The algebra $\mathcal{A}^{\nu-\mathrm{br}}$ also comes with the $*$-structure

$$\omega^\dagger(X) = \overline{\omega(\nu(X)^*)}.$$

From this we can see that $\mathcal{A}^{\nu-\mathrm{br}}$ is the quantum analogue of $\mathcal{O}(H_\nu(G))$.

Indeed the points of $H_\nu(G)$ can be identified with the characters of $\mathcal{O}(G)$ which are compatible with the $*$-structure $\dagger$.

Write $\widetilde{\mathscr{R}}$ for the R-matrix $\mathscr{R} \in \mathcal{D}\hat{\otimes}\mathcal{D}$ seen inside $\mathcal{D}\hat{\otimes}\mathcal{D}_\nu$. Also write

$$\mathscr{R}_\nu := (\nu \otimes \mathrm{id})(\mathscr{R}) = (\mathrm{id} \otimes \nu)(\mathscr{R}).$$

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Then we define the map $I_\nu : \mathcal{A}^{\nu-\mathrm{br}} \to \mathcal{D}_\nu$ given by

$$I_\nu(\omega) := (\omega \otimes \mathrm{id})(\widetilde{\mathscr{R}}_\nu^* \widetilde{\mathscr{R}}).$$

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Theorem

The map $I_\nu : \mathcal{A}^{\nu-\mathrm{br}} \to \mathcal{D}_\nu$ is injective and is a morphism of left Yetter-Drinfeld $\mathcal{D}$-module $*$-algebras.

The image can be characterized: there is an algebra $\mathcal{U}_\nu \subseteq \mathcal{D}_\nu$ given by generators and relations such that $I_\nu(\mathcal{A}^{\nu-\mathrm{br}}) = \mathcal{U}_{\nu,\mathrm{fin}}$.

Moreover $\mathcal{U}_\nu$ is isomorphic to a localization of $\mathcal{A}^{\nu-\mathrm{br}}$.

To proceed we need K-matrices (discussed by various authors).

Definition

We say that $\mathcal{K} \in M(\mathcal{U})$ is a universal K-matrix for $(\mathscr{R}, \mathscr{R}_\nu)$ if

$$\Delta(\mathcal{K}) = \mathscr{R}^{-1}(\mathcal{K} \otimes 1)\mathscr{R}_\nu(1 \otimes \mathcal{K}) = (1 \otimes \mathcal{K})\mathscr{R}_\nu^{-1}(\mathcal{K} \otimes 1)\mathscr{R},$$

together with the condition $\mathcal{K}^* = \nu(\mathcal{K})$.

Remark

For $\nu = \mathrm{id}$ we have that $\mathcal{K} = 1$ is a universal K-matrix for $(\mathscr{R}, \mathscr{R})$.

Generalization of results of [Kolb, Stokman (09)].

Theorem

There is one-to-one correspondence between:

1. *universal K-matrices* $\mathcal{K} \in M(\mathcal{U})$,
2. $*$*-characters* $f : \mathcal{A}^{\nu-\mathrm{br}} \to \mathbb{C}$,
3. $*$*-homomorphisms* $\phi : \mathcal{A}^{\nu-\mathrm{br}} \to \mathcal{A}$ *intertwining* α_ν *with* Δ,
4. $*$*-homomorphisms* $\widehat{\phi} : \mathcal{A}^{\nu-\mathrm{br}} \to \mathcal{U}$ *intertwining* γ_ν *with* Δ.

The correspondence is determined by

$$f(\omega) = \omega(\mathcal{K}), \quad \phi(\omega) = (f \otimes \mathrm{id})\alpha_\nu(\omega), \quad \widehat{\phi}(\omega) = (\mathrm{id} \otimes f)\gamma_\nu(\omega).$$

Explicitely we have the maps

$$\phi(\omega)(X) = \omega(\nu(S(X_{(1)}))\mathcal{K}X_{(2)}), \quad \widehat{\phi}(\omega) = (\mathrm{id} \otimes \omega)(\mathscr{R}_\nu(1 \otimes \mathcal{K})\mathscr{R}^*).$$

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We have that $\mathcal{B} := \phi(\mathcal{A}^{\nu-\mathrm{br}})$ is a right coideal $*$-subalgebra of $\mathcal{A}$, while $\mathcal{C} := \widehat{\phi}(\mathcal{A}^{\nu-\mathrm{br}})$ is a left coideal $*$-subalgebra of $\mathcal{U}$.

$$\begin{tikzcd}
 & \mathcal{A}^{\nu-\mathrm{br}} \arrow[dl, "\phi"'] \arrow[dr, "\widehat{\phi}"] & \\
\mathcal{B} \subseteq \mathcal{A} & & \mathcal{C} \subseteq \mathcal{U}
\end{tikzcd}$$

Remark

For $\nu = \mathrm{id}$ and $\mathcal{K} = 1$ we get $\mathcal{B} = \mathbb{C}1$ and $\mathcal{C} = \mathcal{U}_{\mathrm{fin}}$.

Let us also consider the algebra

$$\mathcal{A}^{\mathcal{C}} := \{\omega \in \mathcal{A} : \omega \triangleleft X = \varepsilon(X)\omega, \ \forall X \in \mathcal{C}\}.$$

Proposition

1. *We have* $\mathcal{K}X = \nu(X)\mathcal{K}$ *for all* $X \in \mathcal{C}$.
2. *We have the inclusion* $\mathcal{B} \subseteq \mathcal{A}^{\mathcal{C}}$.

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Proposition

1. *We have $\mathcal{K}X = \nu(X)\mathcal{K}$ for all $X \in \mathcal{C}$.*
2. *We have the inclusion $\mathcal{B} \subseteq \mathcal{A}^{\mathcal{C}}$.*

Hence by this construction we get the following maps:

$$\mathcal{U}_{\nu,\mathrm{fin}} \xrightarrow{I_\nu^{-1}} \mathcal{A}^{\nu-\mathrm{br}} \xrightarrow{\phi} \mathcal{B} \longrightarrow \mathcal{A}^{\mathcal{C}}.$$

Can we actually find such an element $\mathcal{K}$?

A quantization of the Satake form $\theta \rightsquigarrow \theta_q$ was given by [Letzter (99)]. These are algebra automorphisms of $U_q(\mathfrak{g})$, but not involutions.

Corresponding to them there are coideal subalgebras of $\mathcal{U}$.

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Corresponding to them there are coideal subalgebras of $\mathcal{U}$.

Revisited and extended by [Kolb (14)], which we follow for conventions. We have right coideal subalgebras

$$B_{\mathbf{c},\mathbf{s}} \subseteq U_q(\mathfrak{g}), \quad \Delta(B_{\mathbf{c},\mathbf{s}}) \subseteq B_{\mathbf{c},\mathbf{s}} \otimes U_q(\mathfrak{g}).$$

Here $(\mathbf{c}, \mathbf{s})$ are parameters that need to satisfy certain conditions.

The algebra $B_{\mathbf{c},\mathbf{s}}$ specializes to $U(\mathfrak{g}^\theta)$ in the classical limit.

A universal K-matrix for $B_{\mathbf{c},\mathbf{s}}$ was recently constructed by Balagović and Kolb, generalizing results of Bao and Wang.

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Theorem (Balagović, Kolb (16))

There exists an element $\mathcal{K} \in M(\mathcal{U})$ such that:

$$\mathcal{K} b = \tau\tau_0(b)\mathcal{K}, \quad \forall b \in B_{\mathbf{c},\mathbf{s}},$$
$$\Delta(\mathcal{K}) = \mathscr{R}_{21}(1 \otimes \mathcal{K})\mathscr{R}_{\tau\tau_0}(\mathcal{K} \otimes 1).$$

Here τ_0 is the automorphism corresponding to the longest word of the Weyl group. Note that $\mathscr{R}$ appears in a different way.

However the additional conditions for the parameters $(\mathbf{c}, \mathbf{s})$ are incompatible with the $*$-invariance of the coideal $B_{\mathbf{c},\mathbf{s}}$.

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This can be fixed as follows. We take some special parameters $(\mathbf{c}, \mathbf{s})$ satisfying the extra conditions. Then we define

$$\widetilde{B_{\mathbf{c},\mathbf{s}}} := K_{\omega_0} B_{\mathbf{c},\mathbf{s}} K_{\omega_0}^{-1}, \quad \omega_0 := -\frac{1}{2}(\rho - \rho_X).$$

Lemma (DCNTY (18))

The algebra $\widetilde{B_{\mathbf{c},\mathbf{s}}}$ is $*$-invariant.

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Lemma (DCNTY (18))

The algebra $\widetilde{B_{\mathbf{c},\mathbf{s}}}$ is $*$-invariant.

We can twist the K-matrix in the same way, that is we set

$$\widetilde{\mathcal{K}} := K_{\omega_0} \mathcal{K} K_{\omega_0}^{-1}.$$

Then $\widetilde{\mathcal{K}}$ satisfies similar conditions to $\mathcal{K}$.

We want to compute $\widetilde{\mathcal{K}}^*$. Given a $\mathcal{U}$-module V define

$$\mathcal{S}_0 v := (-1)^{(2\rho^\vee,\lambda)} v, \quad \mathcal{S}_X v := (-1)^{(2\rho_X^\vee,\lambda)} v, \quad v \in V_\lambda.$$

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Theorem

The element $\widetilde{\mathcal{K}}$ satisfies

$$\widetilde{\mathcal{K}}^* = \tau\tau_0(\widetilde{\mathcal{K}}) \circ \mathcal{S}_0 \circ \mathcal{S}_X \circ z \circ z_\tau^{-1}.$$

The term $\mathcal{S}_0 \circ \mathcal{S}_X \circ z \circ z_\tau^{-1}$ contains the additional information needed to relate the Satake form θ to the Vogan form ν.

Let us see how to relate the Satake and Vogan form. We can rewrite

$$\begin{aligned}\theta &= \mathrm{Ad}(z) \circ \tau \circ \omega \circ \mathrm{Ad}(m_X) \\ &= \mathrm{Ad}(z) \circ \tau \circ \tau_0 \circ \mathrm{Ad}(m_0) \circ \mathrm{Ad}(m_X).\end{aligned}$$

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Suppose $\theta(X, \tau, z)$ is inner conjugate to $\nu(Y, \mu)$. The outer part of the automorphism θ is given by $\tau \circ \tau_0$, hence we must have

$$\mu = \tau \circ \tau_0.$$

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To compare signs we give the following definition.

Definition

A sign function $\epsilon : I \to \{\pm 1\}$ is (X, τ)-admissible if it is $\tau\tau_0$-invariant and such that $\nu(Y_\epsilon, \tau\tau_0)$ is inner conjugate to $\theta(X, \tau, z)$.

Any sign function ϵ extends to a group homomorphism $\epsilon : Q \to \mathbb{C}^\times$.

To obtain the Vogan form we look for an extension $\tilde{\epsilon} : P \to \mathbb{C}^\times$ (non-unique) satisfying a certain condition.

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Theorem

Let $\theta = \theta(X, \tau, z)$ be the Satake form. Let ϵ be an (X, τ)-admissible sign function. Then there exists an extension $\tilde{\epsilon} : P \to \mathbb{C}^\times$ of ϵ such that

$$\tilde{\epsilon} \circ \tau\tau_0(\tilde{\epsilon}) = \mathcal{S}_0 \circ \mathcal{S}_X \circ z \circ z_\tau^{-1}.$$

Remark

The theorem does not hold if we replace inner conjugacy with conjugacy! This happens in the case of the real form $\mathfrak{so}^*(4p)$.

Define now the Hopf $*$-algebra automorphism

$$\nu(X) := \tilde{\epsilon}^* \circ \tau\tau_0(X) \circ \tilde{\epsilon}.$$

Here $\tilde{\epsilon}$ acts on a $\mathcal{U}$-module V by $\tilde{\epsilon}v = \tilde{\epsilon}(\lambda)v$, where $v \in V_\lambda$.

This corresponds to the Vogan form of θ, that is

$$\nu(K_i) = K_{\tau\tau_0(i)}, \quad \nu(E_i) = \epsilon_i E_{\tau\tau_0(i)}, \quad \nu(F_i) = \epsilon_i F_{\tau\tau_0(i)}.$$

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$$\nu(K_i) = K_{\tau\tau_0(i)}, \quad \nu(E_i) = \epsilon_i E_{\tau\tau_0(i)}, \quad \nu(F_i) = \epsilon_i F_{\tau\tau_0(i)}.$$

Theorem

The element $\widetilde{\mathcal{K}}' := \widetilde{\mathcal{K}} \circ \tilde{\epsilon}$ satisfies

$$\Delta(\widetilde{\mathcal{K}}') = \mathscr{R}_{21}(1 \otimes \widetilde{\mathcal{K}}')\mathscr{R}_\nu(\widetilde{\mathcal{K}}' \otimes 1),$$

and the condition $\widetilde{\mathcal{K}}'^* = \nu(\widetilde{\mathcal{K}}')$.

This is almost the form we want for the K-matrix.

Let R be the unitary antipode and C the ribbon element of $\mathcal{U}$.

Corollary

Define $\mathcal{K} := C \circ R(\widetilde{\mathcal{K}}')^*$. Then $\mathcal{K}$ is a universal K-matrix, that is

$$\Delta(\mathcal{K}) = \mathscr{R}^{-1}(\mathcal{K} \otimes 1)\mathscr{R}_\nu(1 \otimes \mathcal{K}) = (1 \otimes \mathcal{K})\mathscr{R}_\nu^{-1}(\mathcal{K} \otimes 1)\mathscr{R},$$

together with the condition $\mathcal{K}^* = \nu(\mathcal{K})$.

Hence we can use this element $\mathcal{K}$ for the general construction.

We write $\mathcal{U}^\theta := R(\widetilde{B_{\mathbf{c},\mathbf{s}}})$. This is the quantum analogue of $U(\mathfrak{u}^\theta)$.

Recall the definition of the algebra $\mathcal{C} = \widehat{\phi}(\mathcal{A}^{\nu-\mathrm{br}})$.

We write $\mathcal{U}^\theta := R(\widetilde{B_{\mathbf{c},\mathbf{s}}})$. This is the quantum analogue of $U(\mathfrak{u}^\theta)$.

Recall the definition of the algebra $\mathcal{C} = \widehat{\phi}(\mathcal{A}^{\nu-\mathrm{br}})$.

Theorem

We have $\mathcal{C} \subseteq \mathcal{U}^\theta$ and $\mathcal{A}^{\mathcal{C}} = \mathcal{A}^{\mathcal{U}^\theta}$.

Remark

We can think of $\mathcal{C}$ as the "locally-finite" part of $\mathcal{U}^\theta$. Indeed in the case $\nu = \mathrm{id}$ and $\mathcal{K} = 1$ we have $\mathcal{U}^\theta = \mathcal{U}$ and $\mathcal{C} = \mathcal{U}_{\mathrm{fin}}$.

Let us also consider the other algebra

$$\mathcal{B} = \phi(\mathcal{A}^{\nu-\mathrm{br}}) \subseteq \mathcal{A}^{\mathcal{C}} = \mathcal{A}^{\mathcal{U}^{\theta}}.$$

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Theorem

The map $\phi : \mathcal{A}^{\nu-\mathrm{br}} \to \mathcal{A}^{\mathcal{C}}$ is surjective, that is $\mathcal{B} = \mathcal{A}^{\mathcal{U}^\theta}$.

An important part in the proof is played by the following result.

Theorem (Letzter (00))

$\mathcal{A}^{\mathcal{U}^\theta}$ has the same harmonic decomposition as in the classical case.

Putting all together we have the following result.

Theorem

We obtain a surjective $\mathcal{U}$-equivariant $*$-homomorphism

$$\mathcal{U}_{\nu,\mathrm{fin}} \xrightarrow{I_\nu^{-1}} \mathcal{A}^{\nu-\mathrm{br}} \xrightarrow{\phi} \mathcal{A}^{\mathcal{U}^\theta}.$$

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This should be compared with the classical map:

$$(\mathcal{O}(H_\nu(G)), \mathrm{Ad}_\nu^*) \twoheadrightarrow (\mathcal{O}(U/U^\theta), L^*).$$

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Complementary to [De Commer, Neshveyev (15)] for flag manifolds.
Should fit into this framework with appropriate modifications.

Future plan: relate representation categories of the various algebras.