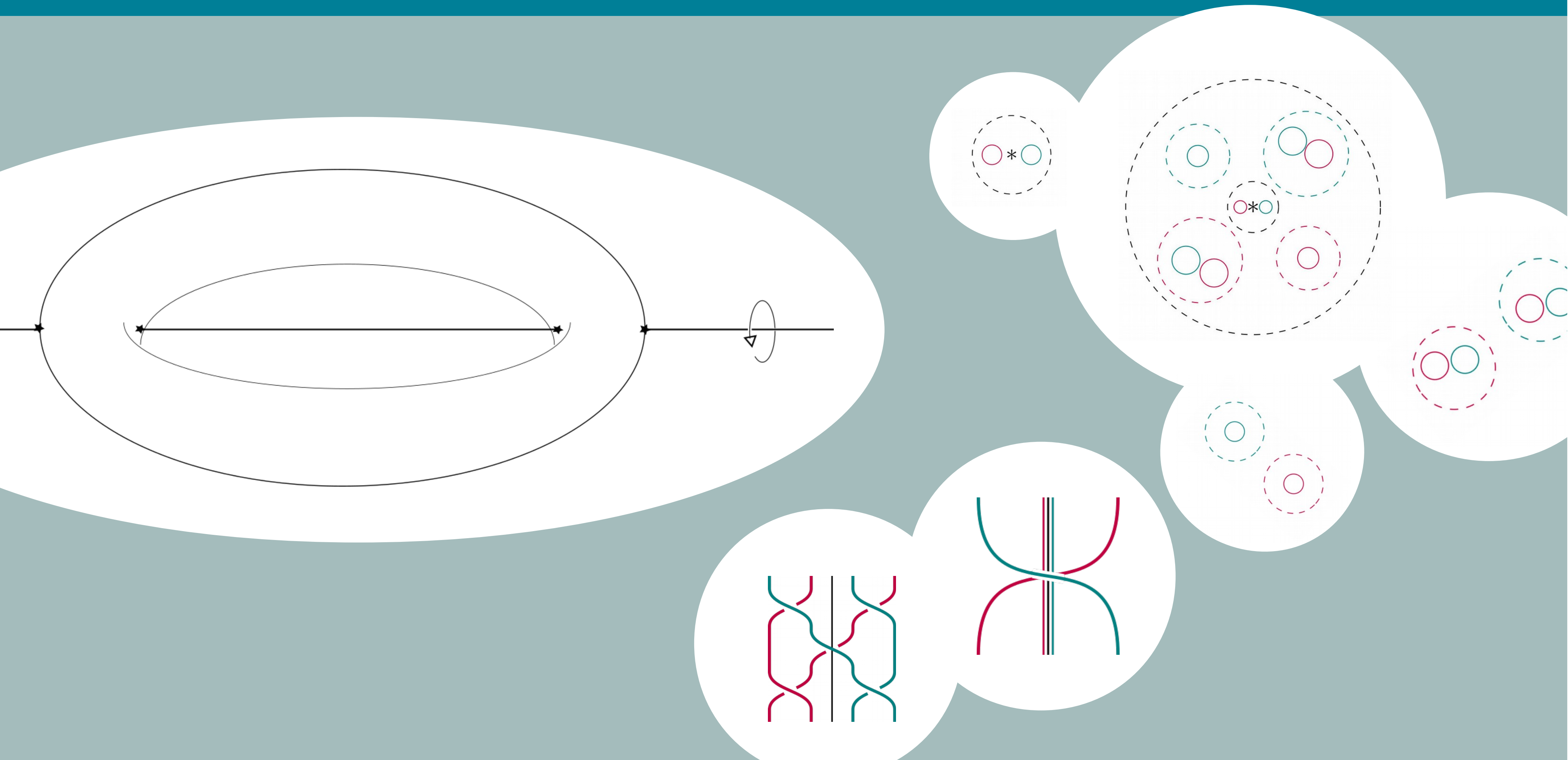
Quantum symmetric pairs, low-dimensional topology & Hecke algebras



Tim Weelinck

THE UNIVERSITY of EDINBURGH

EPSRC Engineering and Physical Sciences Research Council

erc European Research Council

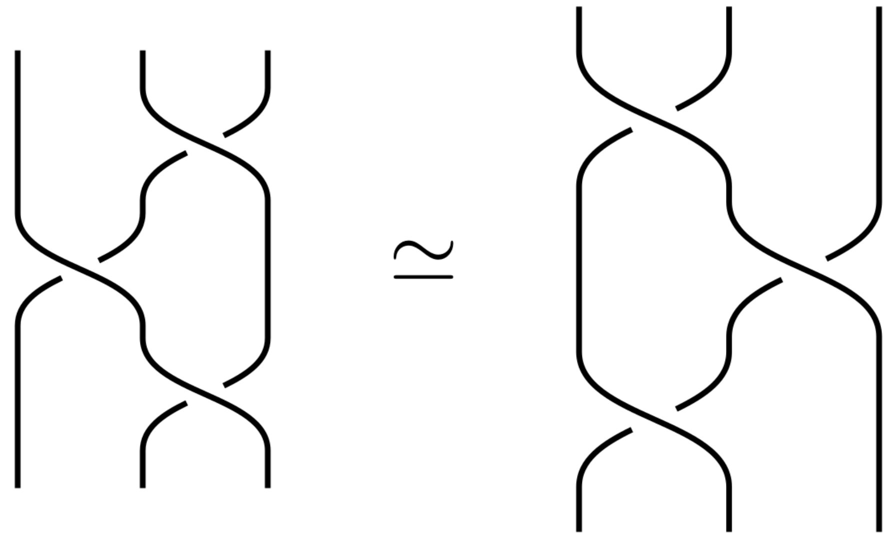
ICMS Quantum Homogeneous Spaces

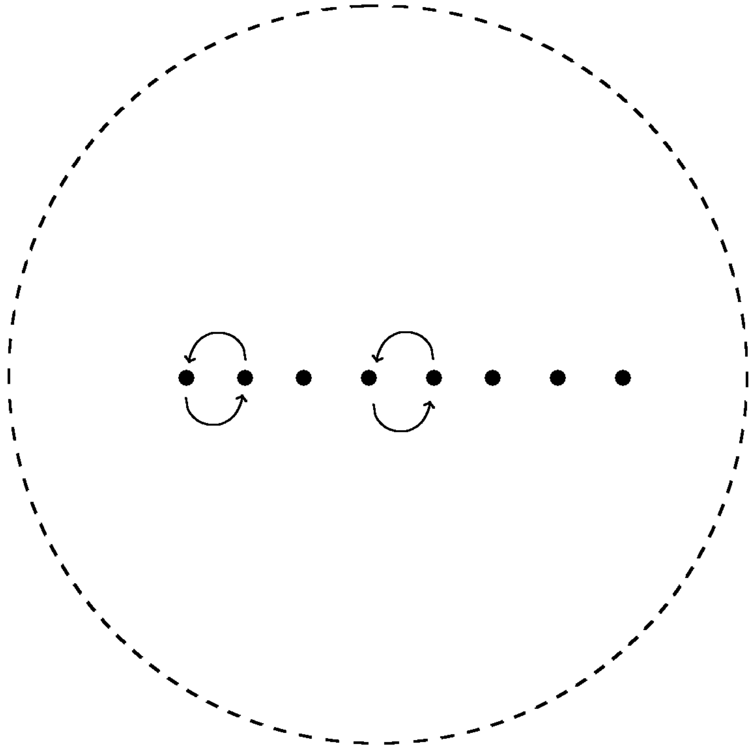
June 14th 2018

The double affine Hecke algebra of type $C^\vee C_n$

The Dynkin diagram of type A

The braid group of type A

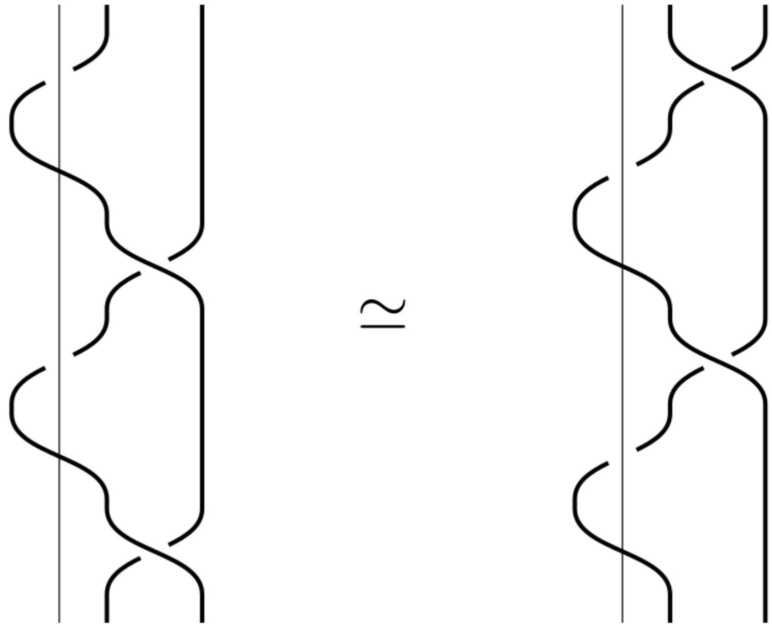


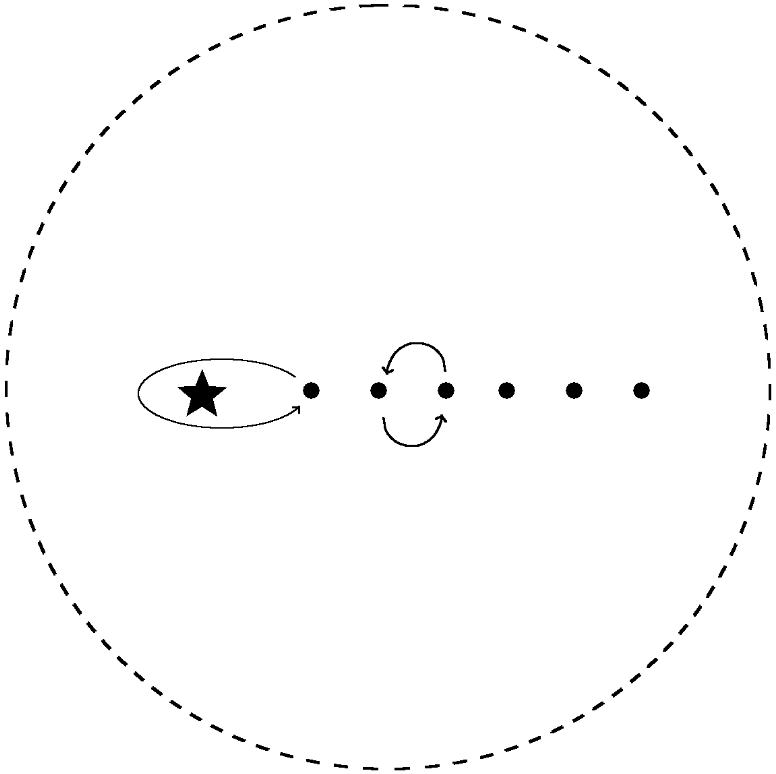


The double affine Hecke algebra of type $C^\vee C_n$

The Dynkin diagram of type B

The braid group of type B

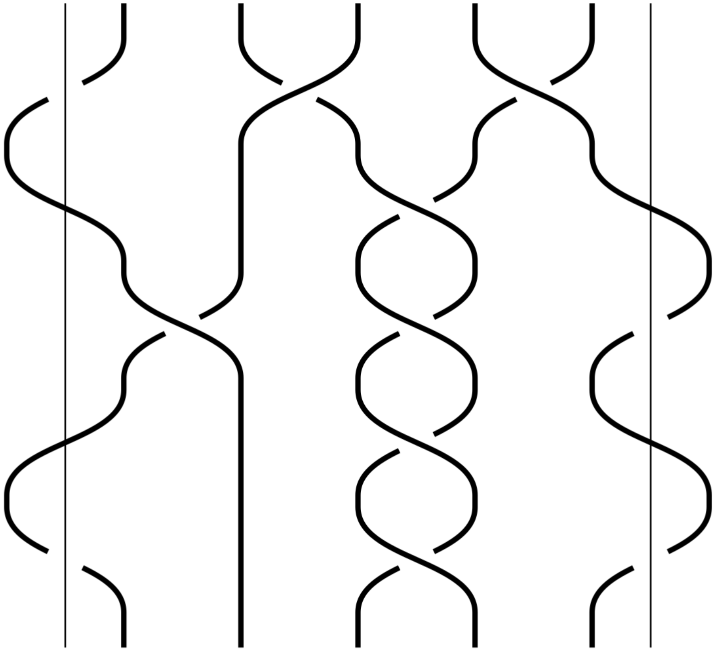


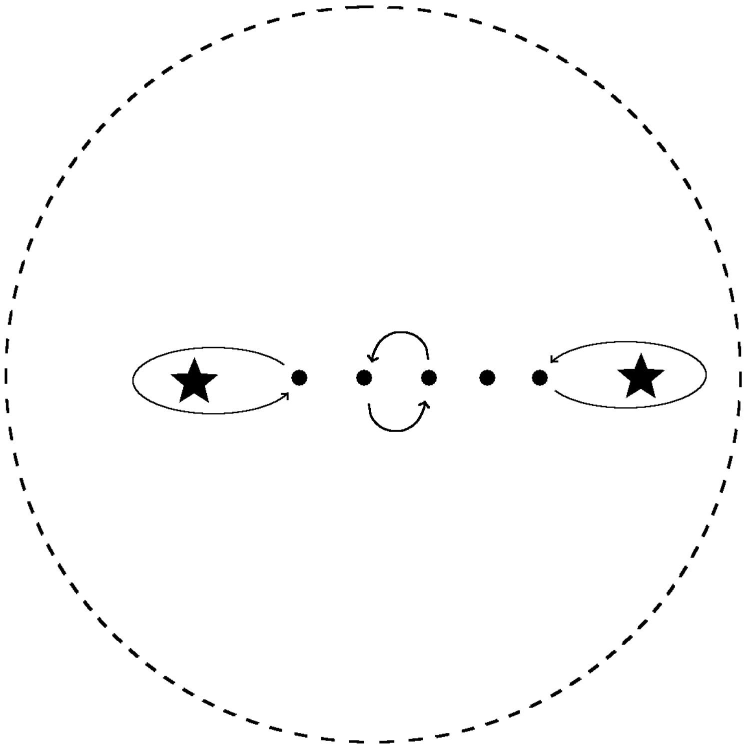


The double affine Hecke algebra of type $C^\vee C_n$

The Affine Dynkin diagram of type $C^\vee C_n$

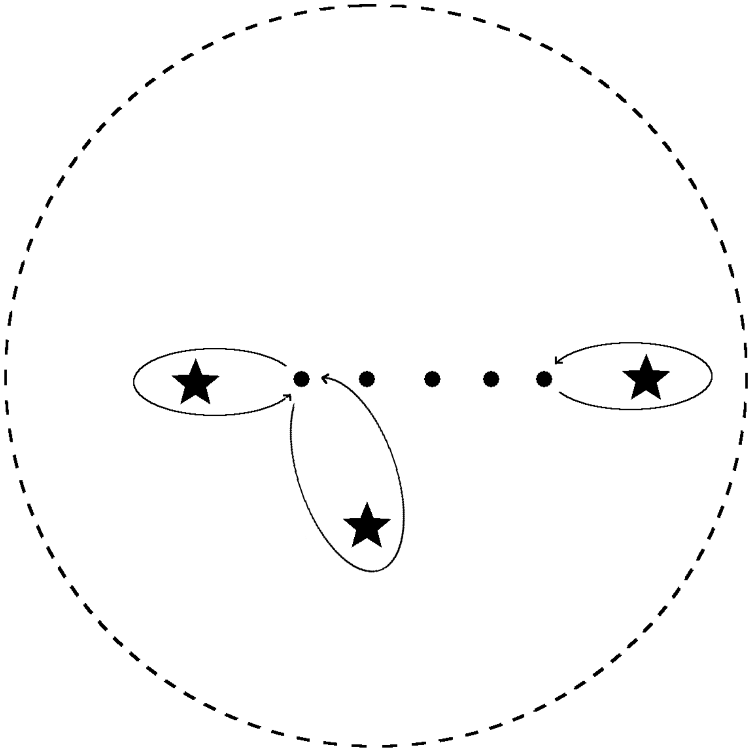
The affine braid group $\widehat{B}_n$





The double affine Hecke algebra of type $C^\vee C_n$

The double affine braid group $\widetilde{B_n}$



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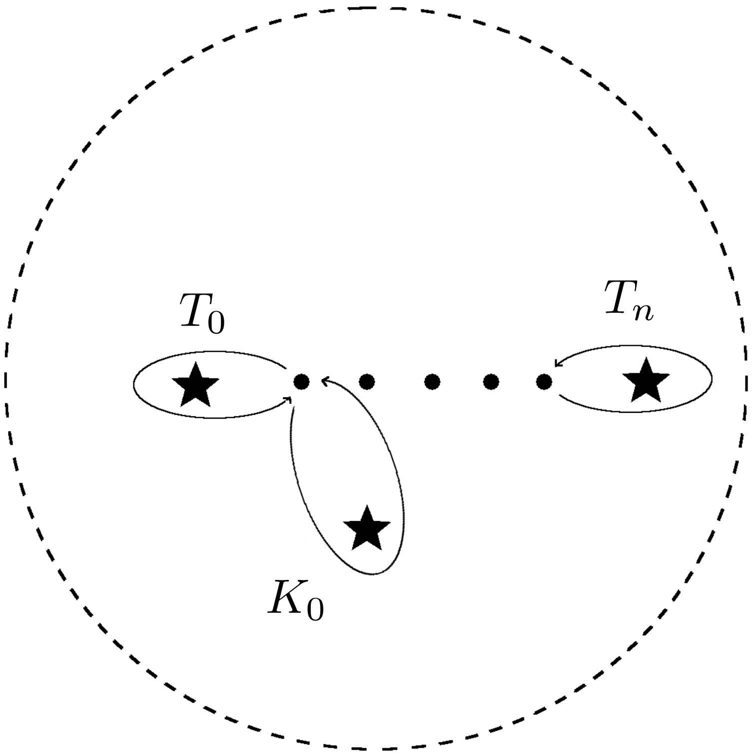

Generators

$T_0, \ldots, T_n, K_0$

The double affine Hecke algebra of type $C^\vee C_n$

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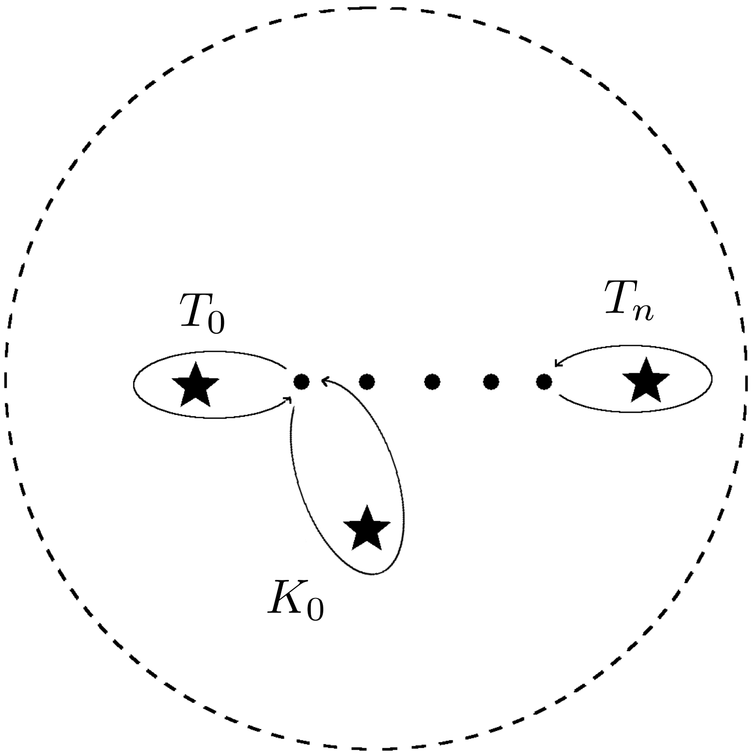

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Type A braid relations (0 < i< n-1)

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},$$

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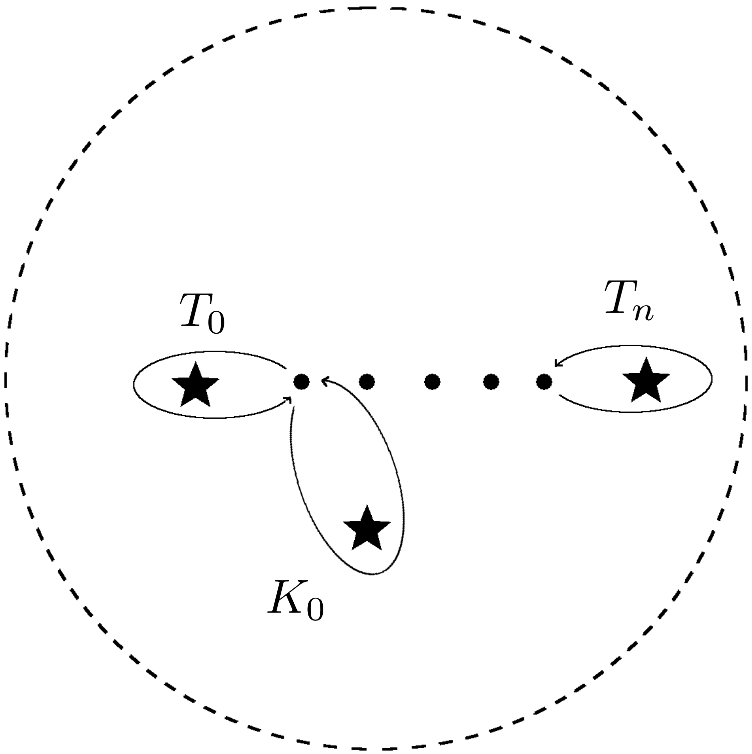

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Type B braid relations

$$T_{n-1} T_n T_{n-1} T_n = T_n T_{n-1} T_n T_{n-1},$$

$$T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0,$$

$$K_0 T_1 K_0 T_1 = T_1 K_0 T_1 K_0,$$

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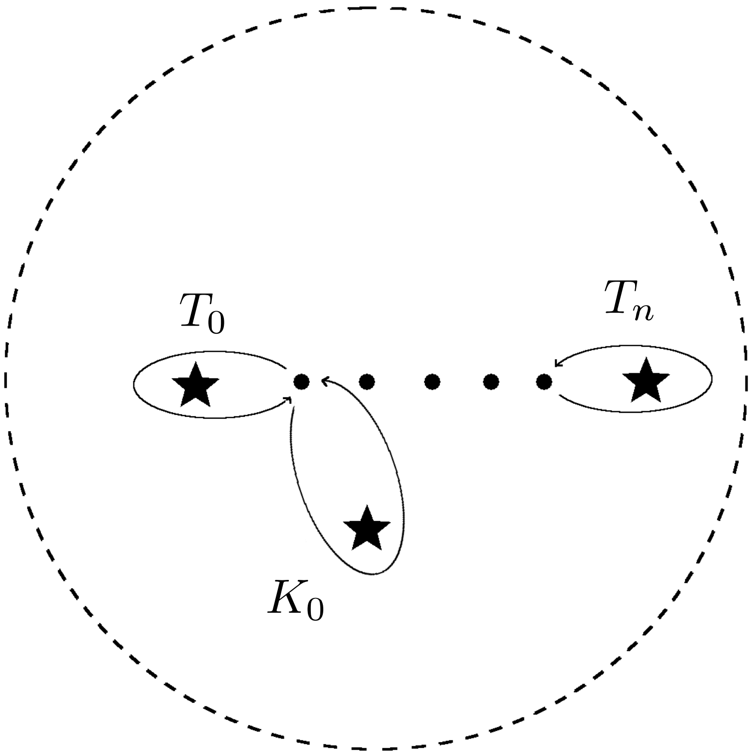

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$$K_0 T_1 K_0 T_1 = T_1 K_0 T_1 K_0,$$

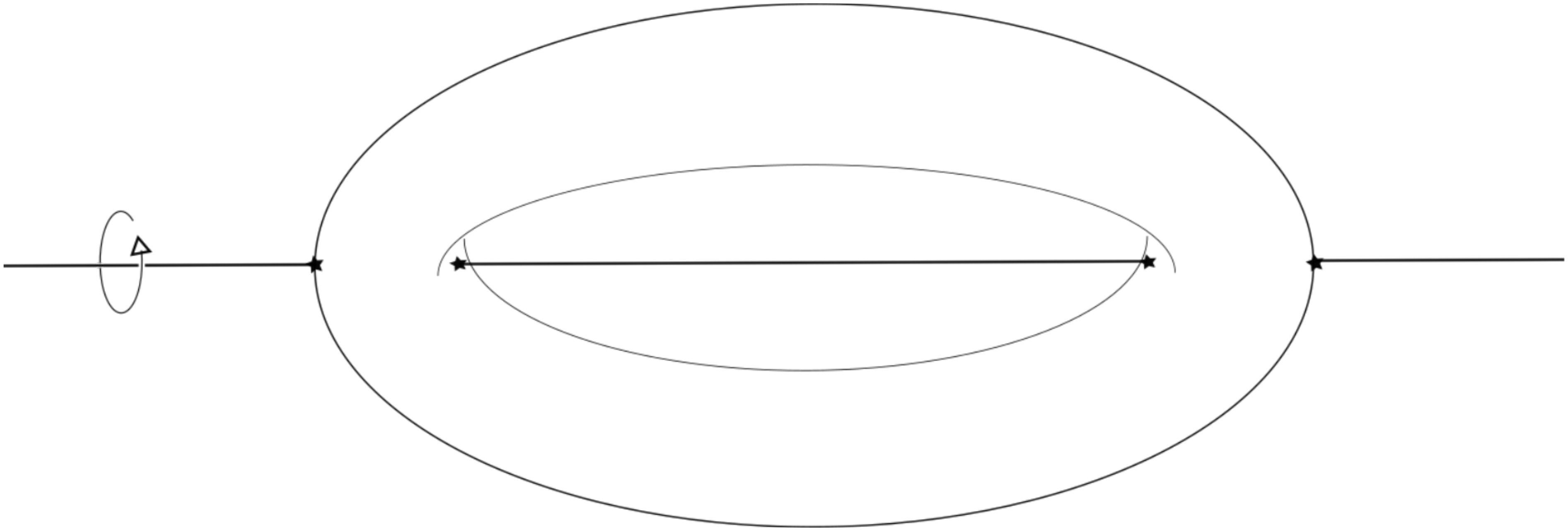
Commutation relations

$$T_i T_j = T_j T_i \text{ if } |i-j| > 1,$$

$$K_0 T_i = T_i K_0 \text{ for } i > 1,$$

$$T_0 T_1^{-1} K_0 T_1 = T_1^{-1} K_0 T_1 T_0,$$

The double affine braid group $\widetilde{B_n}$



$$\widetilde{B_n} = B_n[\mathbb{T}/\mathbb{Z}_2]$$

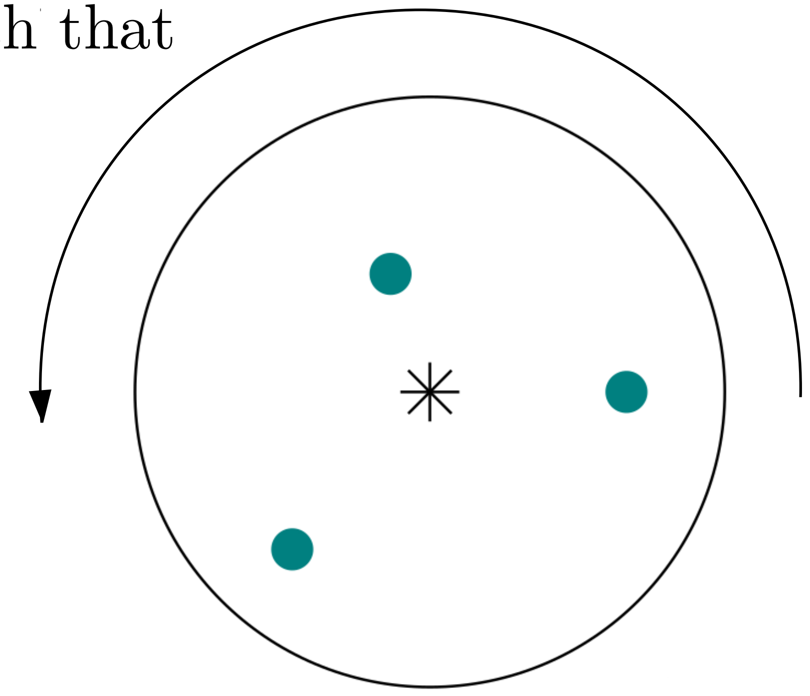
The double affine Hecke algebra of type $C^\vee C_n$

Definition (Folk)

A configurations of points in $[\Sigma/\mathbb{Z}_2]$ is

a configuration of points in Σ such that

Z_2-orbits of points do not collide.



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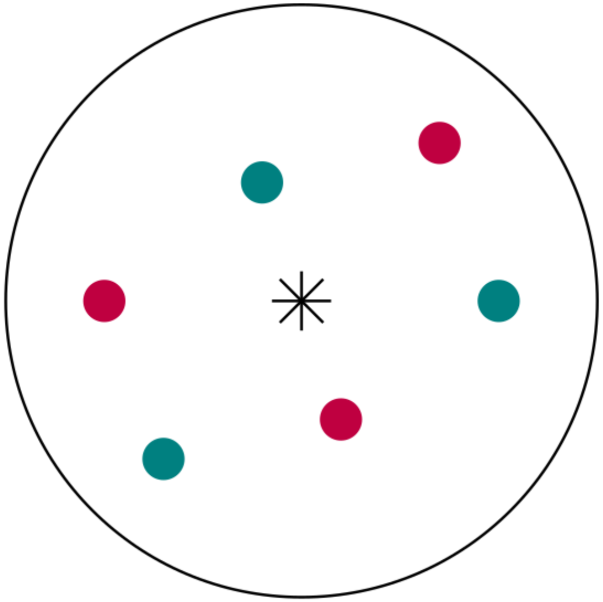
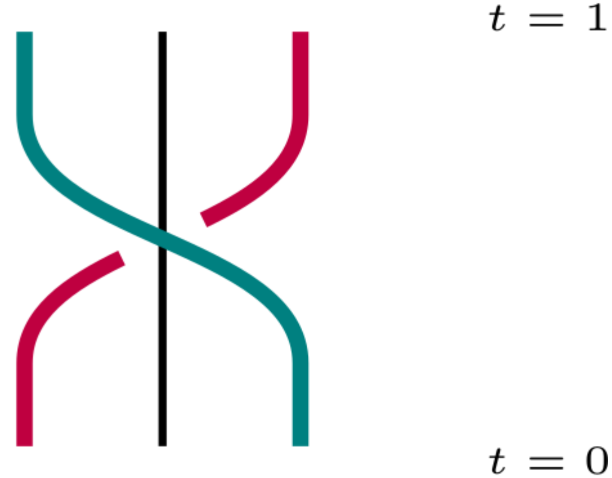


The double affine Hecke algebra of type $C^\vee C_n$

$\rightsquigarrow$



The double affine Hecke algebra of type $C^\vee C_n$

Definition (1999 S. Sahi; 2000 M. Noumi, J. Stokman)

The $C^\vee C_n$ DAHA is a 6-parameter Hecke quotient of the group algebra of the double affine braid group e.g.

$$(T_i - t)(T_i + t^{-1}) = 0,$$

$$(K_0 - u)(K_0 + u^{-1}) = 0,$$

and so forth

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and so forth

Theorem (2008 D. Jordan, X. Ma)

One can construct $C^\vee C_n$ DAHA representations out of the following data:

- a quantum D-module,
- the vector representation,
- two quantum symmetric pairs (AIII/AIV),
- two characters of the coideal subalgebras.

Main Result

Theorem (W.)

1. To each quantum symmetric pair there is a uniquely associated two-dimensional Z_2-orbifold TQFT.

2. Any such TQFT produces canonical orbifold braid group actions.

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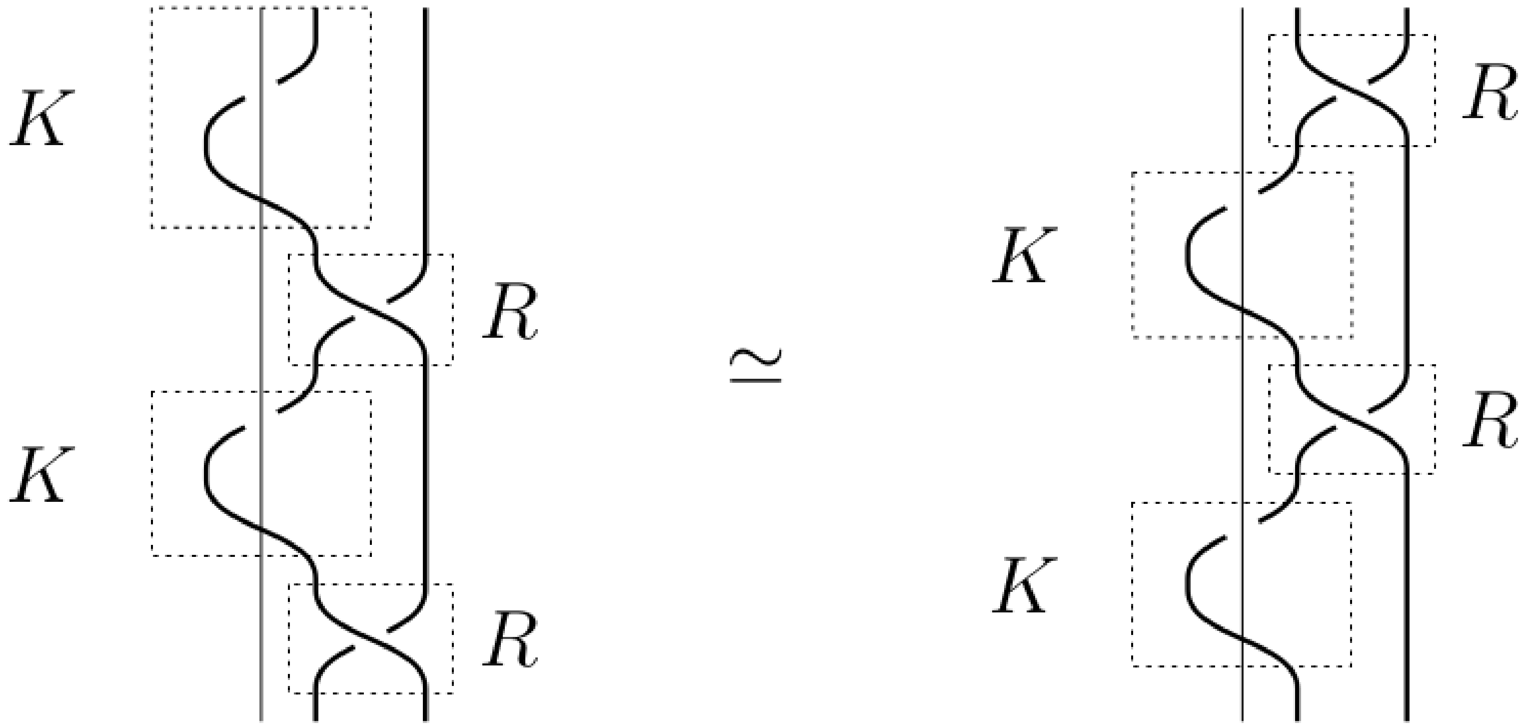
2. Any such TQFT produces canonical orbifold braid group actions.

Hope/expectation

This recovers the DAHA representations of D. Jordan and X. Ma (and therefore extends their results to QSP of any type and higher genus surfaces).

Twisted reflection equations

$$K_{12}\,R_{32}\,K_{13}\,R_{23} \quad = \quad R_{32}\,K_{13}\,R_{23}\,K_{12}$$



Twisted reflection equations

Theorem ('15 M. Balagovic, S. Kolb; '17 Kolb)

Let $(\mathcal{U}_q(\mathfrak{g}), B_{c,s})$ be a quantum symmetric pair in G. Letzter's classification. Then there exists a universal K-matrix $K \in B_{c,s} \otimes \mathcal{U}_q(\mathfrak{g})$

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$$K_{12}\, R_{32}^{\tau\tau_0}\, K_{13}\, R_{23} \quad = \quad R_{32}\, K_{13}\, R_{23}^{\tau\tau_0}\, K_{12}$$

$$\tau\tau_0 : \mathcal{U}_q(\mathfrak{g}) \to \mathcal{U}_q(\mathfrak{g}), \qquad (\tau\tau_0)^2 = \mathrm{id},$$

$$(\tau\tau_0 \otimes \tau\tau_0)(R) = R, \qquad R^{\tau\tau_0} := (\tau\tau_0 \otimes \mathrm{id})(R).$$

Twisted reflection equations

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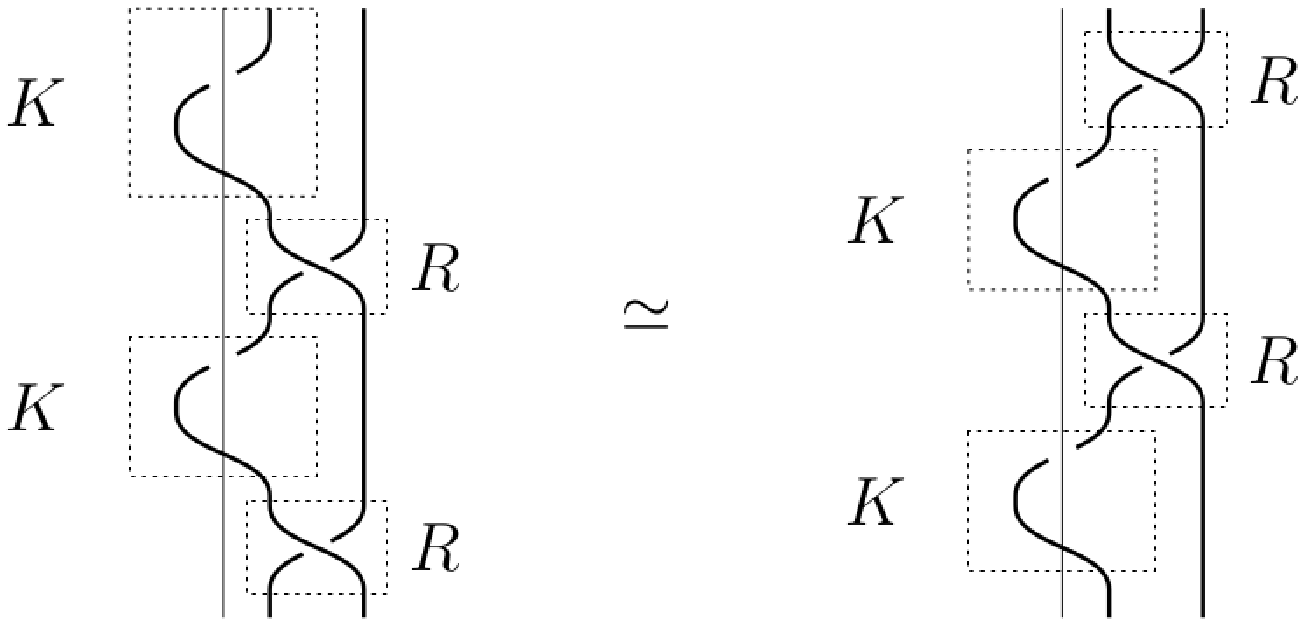

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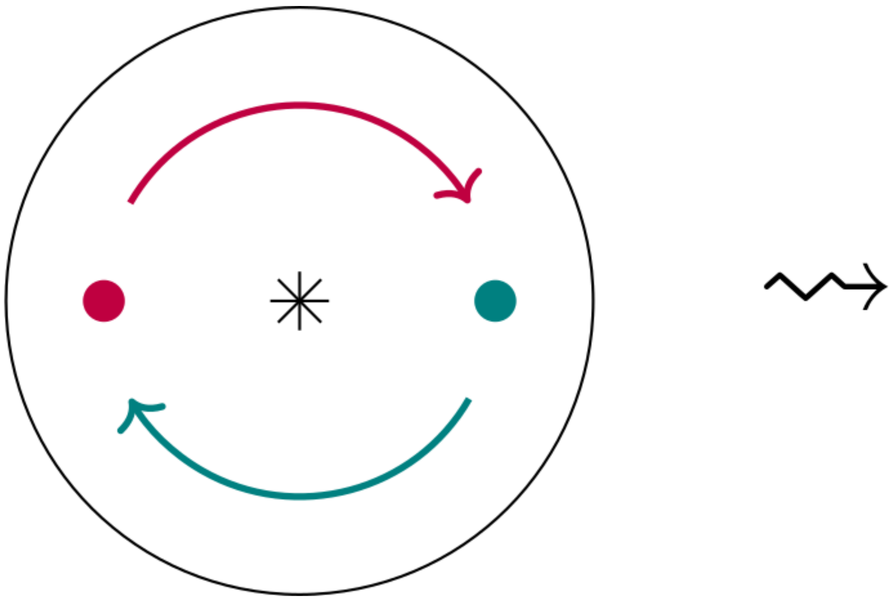

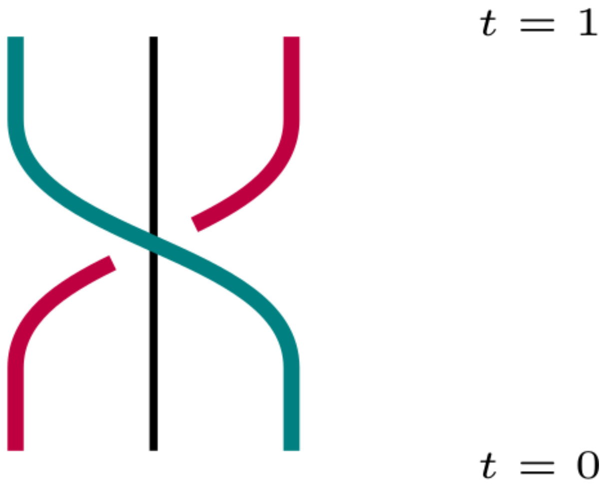

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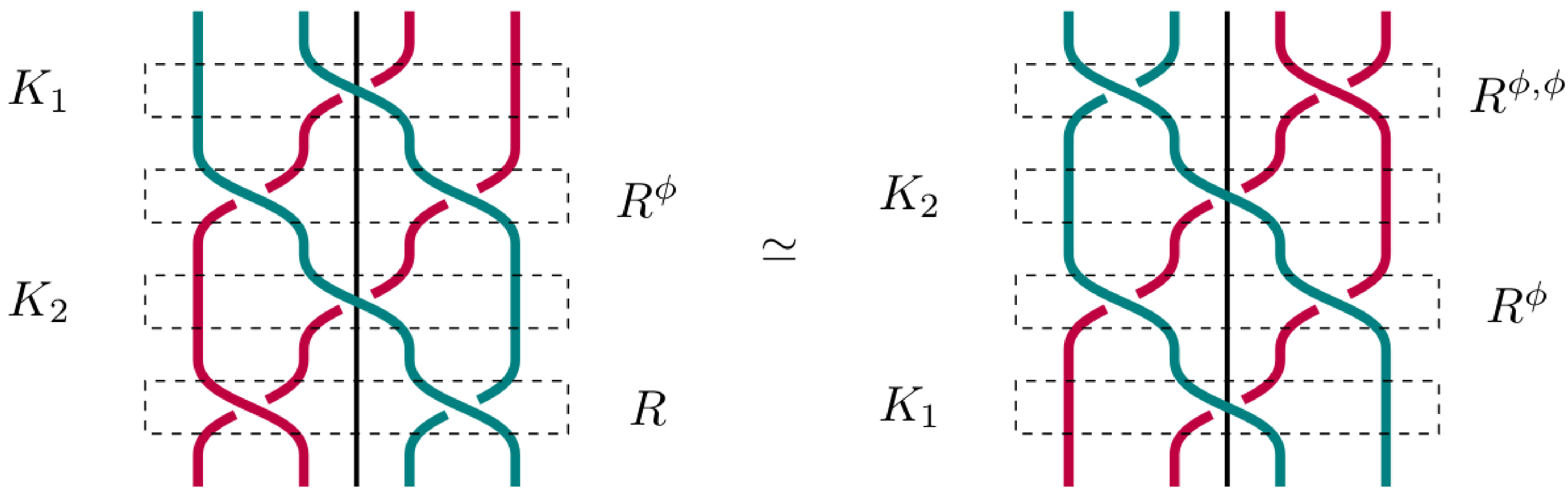
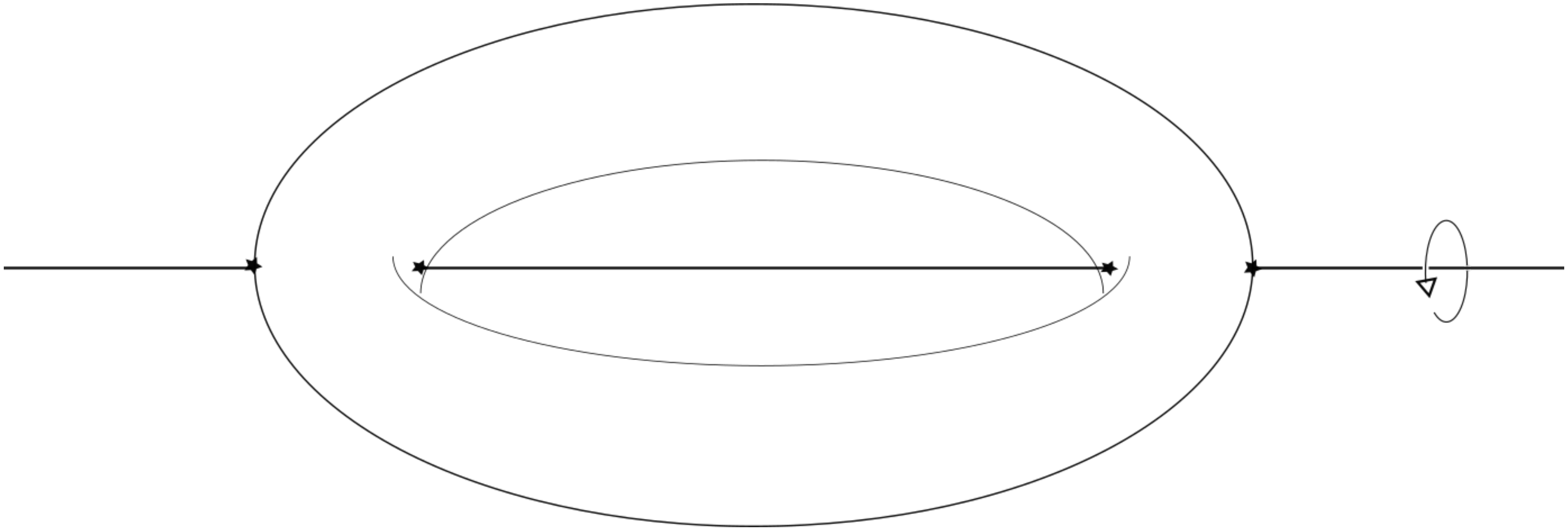


(Imprecise) Definition

$$\mathcal{Z} : \mathbb{Z}_2\text{-}\mathbf{Orb}_2 \to \mathbf{Cat},$$
$$\text{2d } \mathbb{Z}_2\text{-orbifold} \mapsto \text{category},$$



Equivariant topological quantum field theories

(Imprecise) Definition

$$\begin{aligned}\mathcal{Z} : \mathbb{Z}_2\text{-}\mathbf{Orb}_2 &\to \mathbf{Cat},\\ \text{2d } \mathbb{Z}_2\text{-orbifold} &\mapsto \text{category},\\ \text{(equivariant) embedding} &\mapsto \text{functor},\\ \text{isotopy} &\mapsto \text{natural isomorphism}\end{aligned}$$

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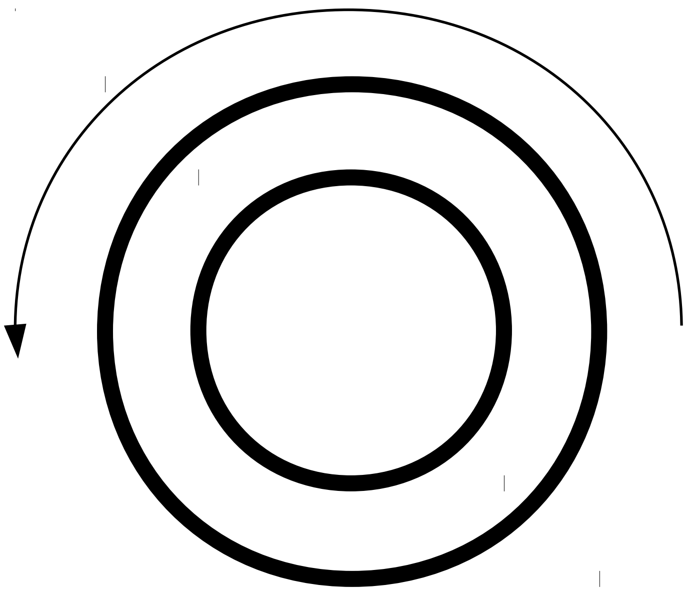
$$\mathcal{Z}(M_1 \underset{N}{\cup} M_2) \cong \mathcal{Z}(M_1) \underset{\mathcal{Z}(N)}{\otimes} \mathcal{Z}(M_2)$$

Example: the orbifold annulus

$\mathcal{Z} = \mathcal{Z}_{\mathcal{U}_q(\mathfrak{g}), B_{c,s}}$

$[\mathrm{Ann}/\mathbb{Z}_2]$



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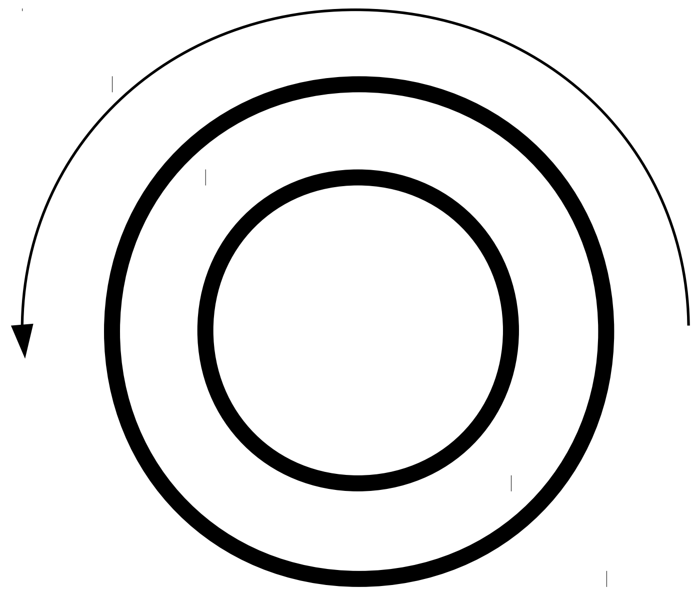

Theorem (W.)

$$\mathcal{Z}[\mathrm{Ann}/\mathbb{Z}_2] \cong \mathcal{O}_q(G)^{\tau\tau_0}\text{-mod}^{\mathcal{U}_q(\mathfrak{g})}$$

Example: the orbifold annulus

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Remark

Here $\mathcal{O}_q(G)^{\tau\tau_0} = \bigoplus X^* \otimes X^{\tau\tau_0}$ is a $\tau\tau_0$-twisted version of the Majid's braided dual (a.k.a. reflection equation algebra).

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Theorem (2002 J. Donin, P. Kulish, A. Mudrov)

$\mathcal{O}_q(G)$ is a universal source of solutions to the reflection equation.

Equivariant topological quantum field theories

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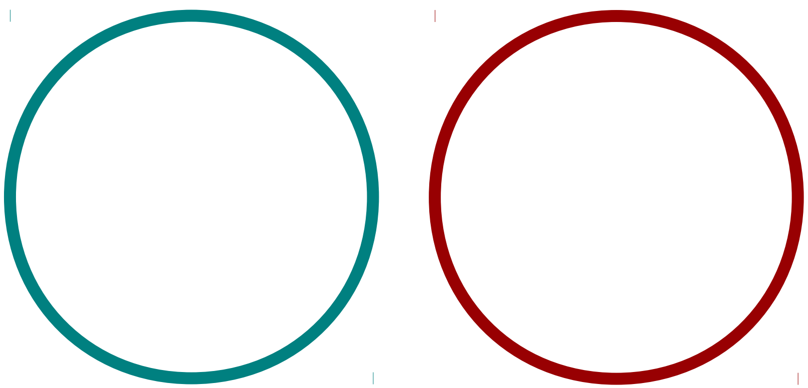
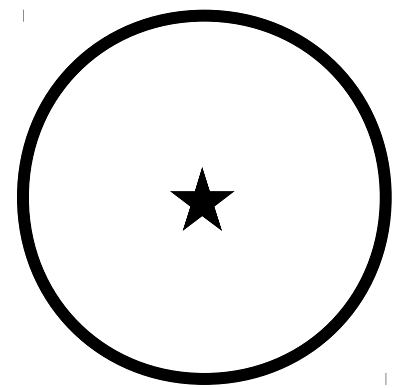
Equivariant topological quantum field theories

Local Observables (orbifolds)

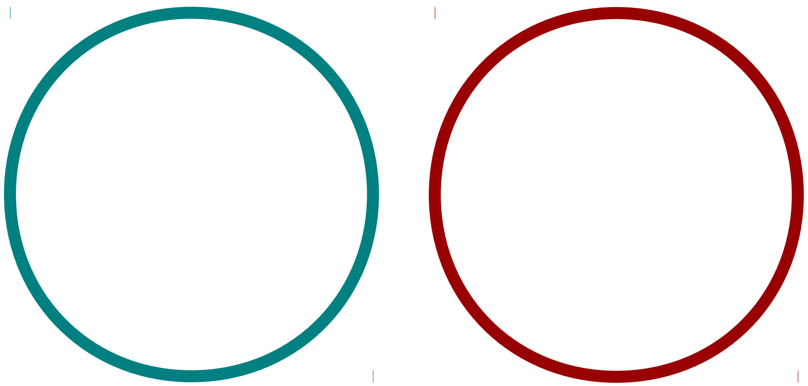


$\mathbb{D}^2$

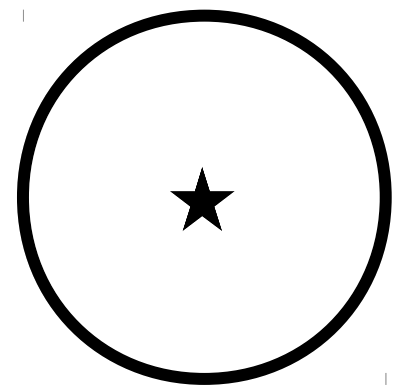


$\mathbb{D}^2_*$

Local Observables (orbifolds)



$$\mathcal{A} = \mathcal{Z}(\mathbb{D}^2)$$



$$\mathcal{M} = \mathcal{Z}(\mathbb{D}^2_*)$$

Equivariant topological quantum field theories

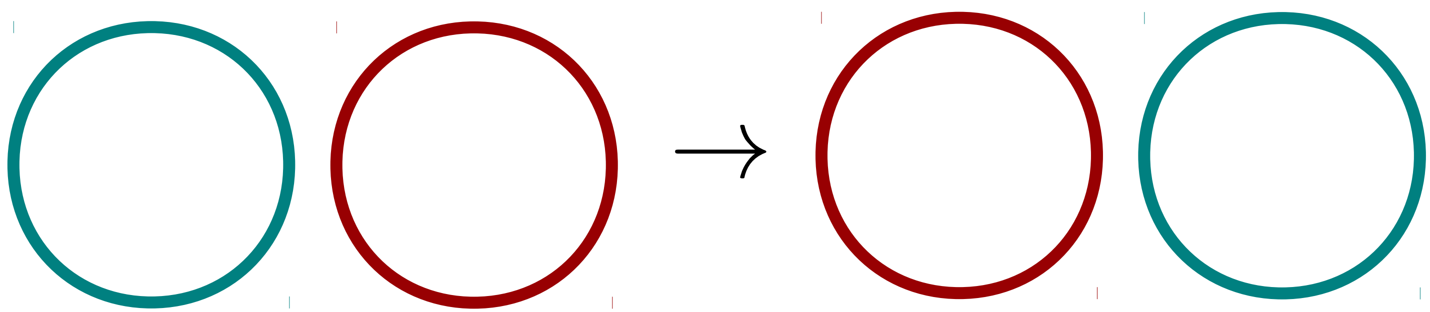
$\mathcal{A} = \mathcal{U}_q(\mathfrak{g})\text{-mod}_{\text{fin.dim.}}$

$\mathcal{M} = B_{c,s}\text{-mod}_{\text{fin.dim.}}$

Local Observables

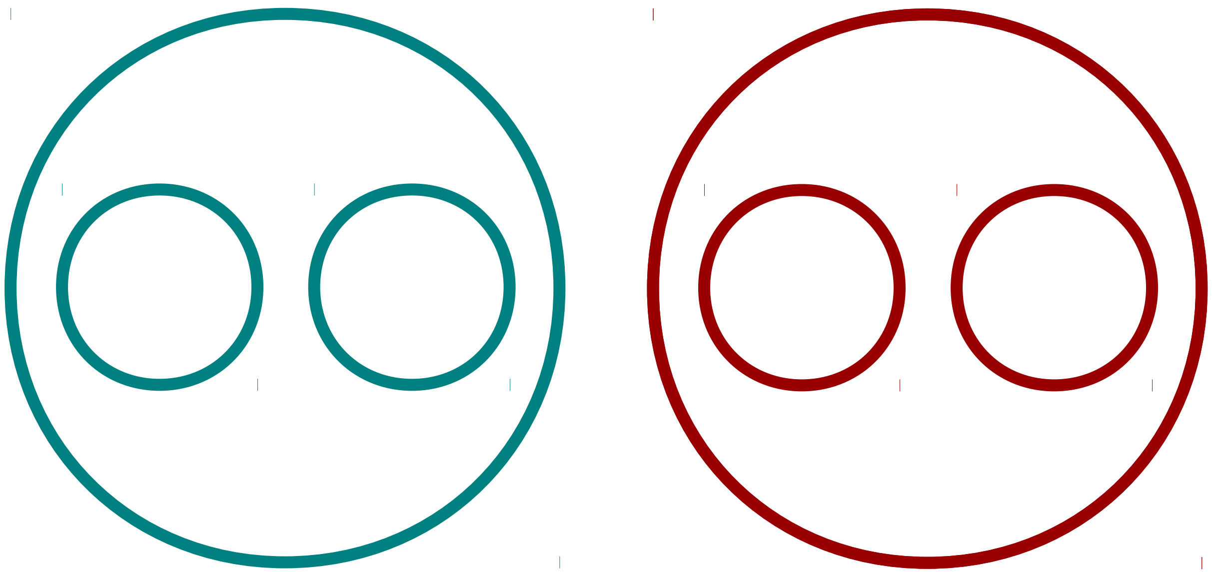
Quantum Symmetric Pair

Local Observables (embeddings)



$$\Phi : \mathcal{A} \to \mathcal{A}$$

Local Observables (embeddings)



$$\otimes : \mathcal{A} \boxtimes \mathcal{A} \to \mathcal{A}$$

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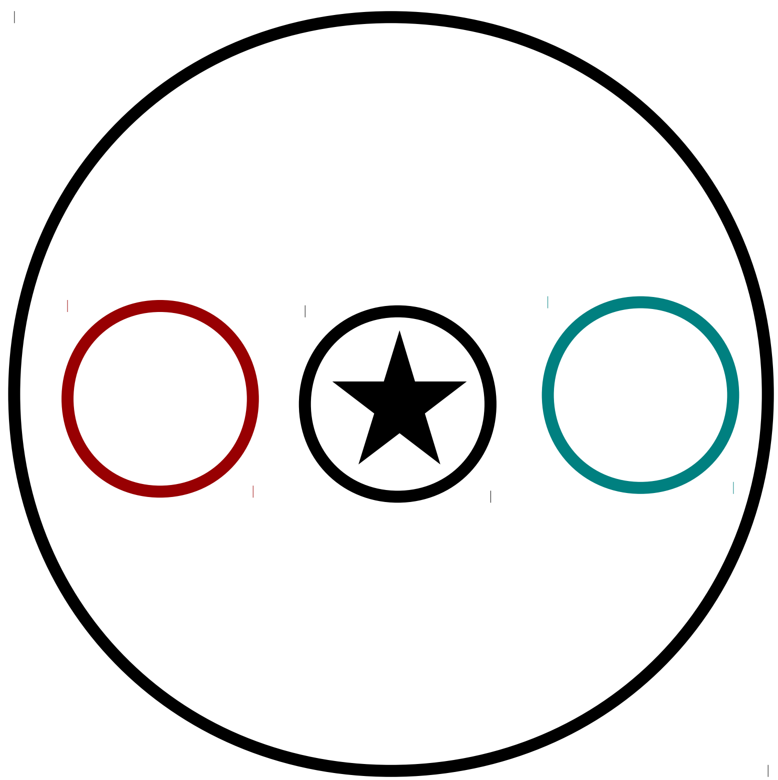


$$\otimes : \mathcal{M} \boxtimes \mathcal{A} \to \mathcal{M}$$

Equivariant topological quantum field theories

$$\mathcal{A} = \mathcal{U}_q(\mathfrak{g})\text{-mod}_{\text{fin.dim.}} \qquad \mathcal{M} = B_{c,s}\text{-mod}_{\text{fin.dim.}}$$

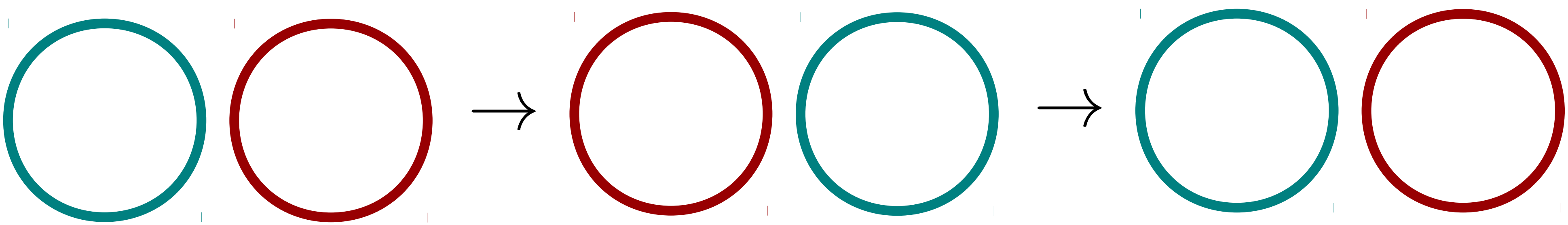
Local Observables

$$\otimes : \mathcal{A} \boxtimes \mathcal{A} \to \mathcal{A}$$
$$\Phi : \mathcal{A} \to \mathcal{A}$$
$$\otimes : \mathcal{M} \boxtimes \mathcal{A} \to \mathcal{A}$$

Quantum Symmetric Pair

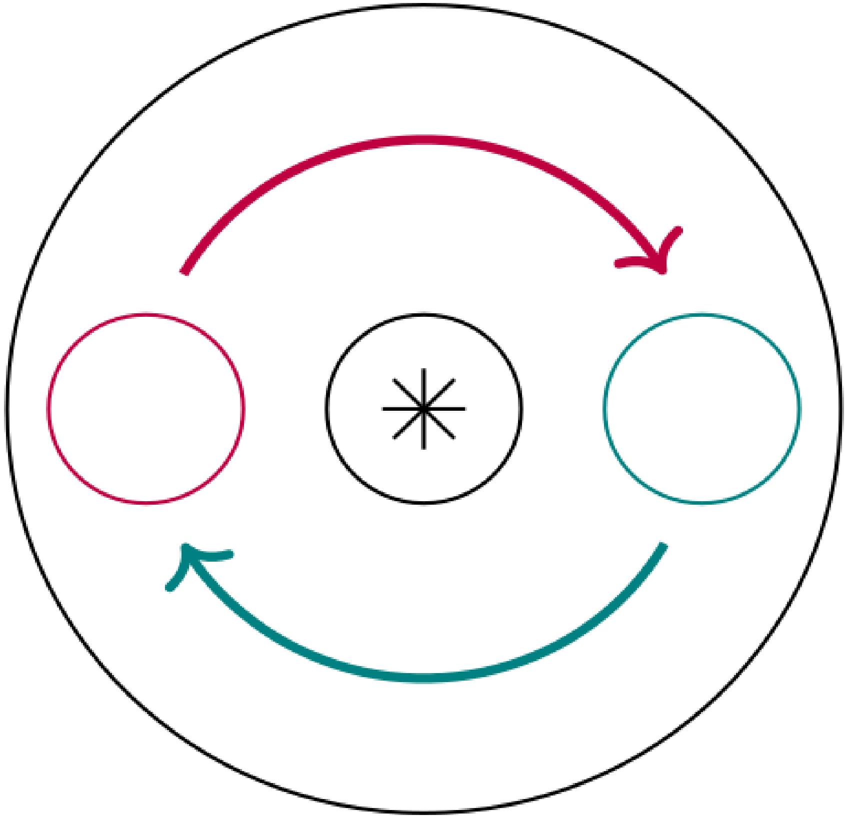
$$\Delta : \mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{U}_q(\mathfrak{g}) \to \mathcal{U}_q(\mathfrak{g})$$
$$\tau\tau_0 : \mathcal{U}_q(\mathfrak{g}) \to \mathcal{U}_q(\mathfrak{g})$$
$$\Delta : B_{c,s} \to B_{c,s} \otimes \mathcal{U}_q(\mathfrak{g})$$

Local Observables (isotopies)



$$\Phi^2 \cong \mathrm{id}_{\mathcal{A}}$$

Local Observables (isotopies)



$$\otimes \cong \otimes \circ (\mathrm{id}_{\mathcal{M}} \boxtimes \Phi)$$

Equivariant topological quantum field theories

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Local Observables

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$\tau\tau_0 : \mathcal{U}_q(\mathfrak{g}) \to \mathcal{U}_q(\mathfrak{g})$

$\Delta : B_{c,s} \to B_{c,s} \otimes \mathcal{U}_q(\mathfrak{g})$

$(\tau\tau_0)^2 = \mathrm{id}_{\mathcal{U}_q(\mathfrak{g})}$

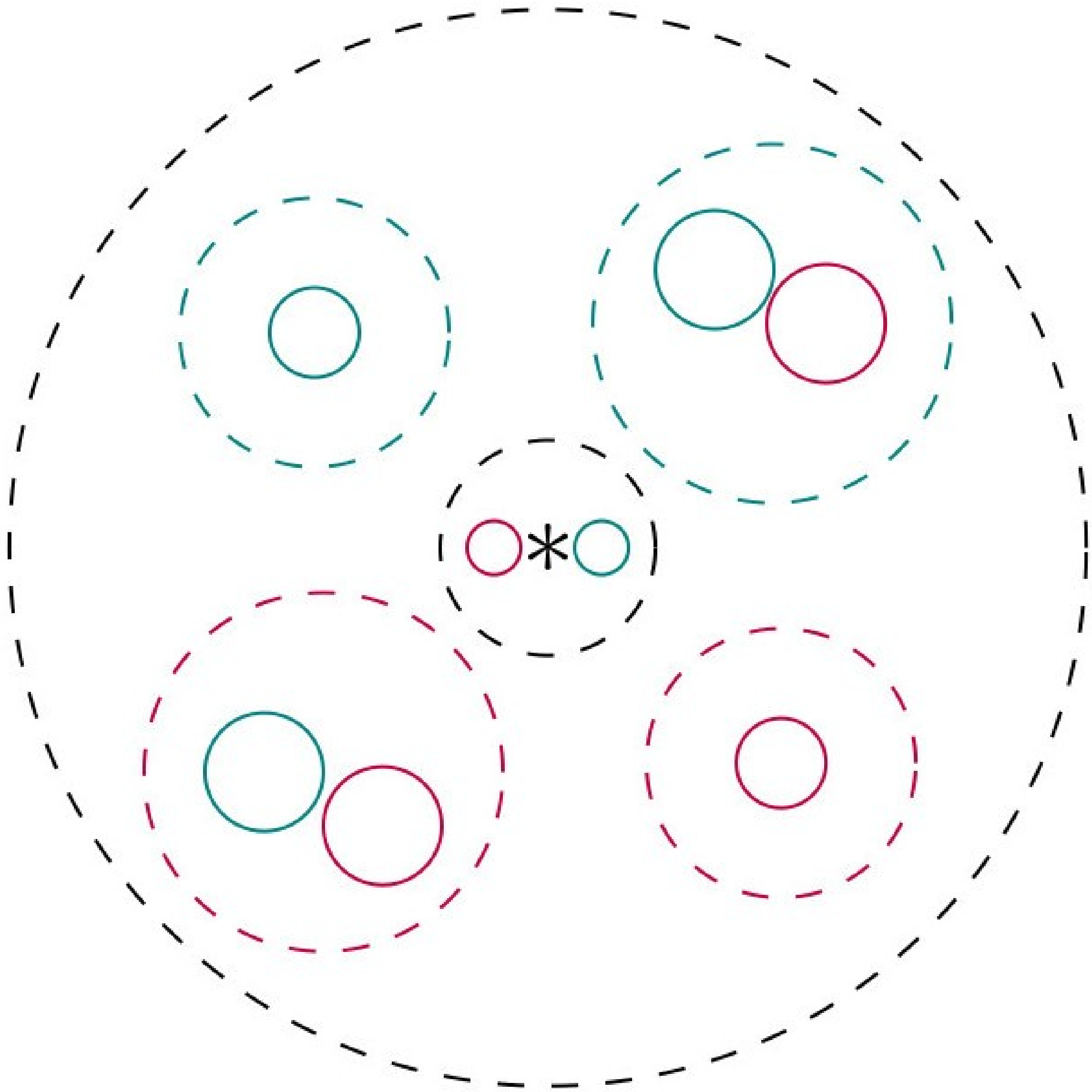
$K\Delta(b) = \mathrm{id} \otimes \tau\tau_0\Delta(b)K$

Equivariant topological quantum field theories

Theorem (W.) arXiv:1804.02315

For any quantum symmetric pair, their categories of modules define local observables of a 2-dimensional Z_2-orbifold TQFT.

Equivariant topological quantum field theories



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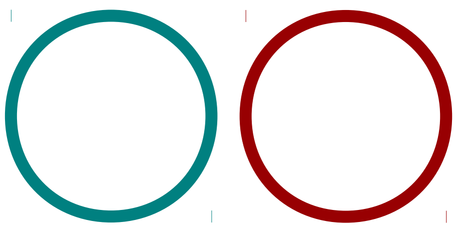
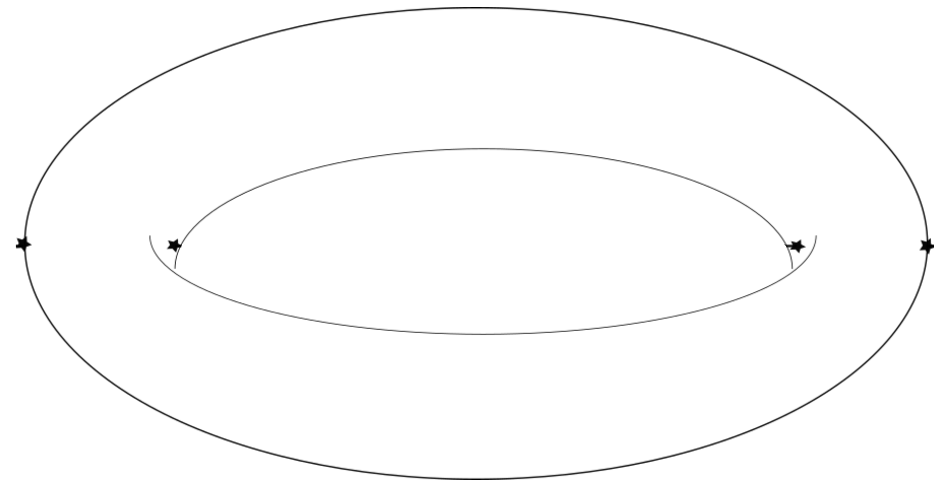
Corollary

Any quantum symmetric pair has a uniquely associated two-dimensional Z_2-orbifold TQFT.

Orbifold braid group actions

$f : \coprod_n$  $\hookrightarrow$ 

$$\mathcal{Z}(f) : \mathcal{A} \boxtimes \mathcal{A} \cdots \boxtimes \mathcal{A} \to \mathcal{Z}[\mathbb{T}/\mathbb{Z}_2]$$

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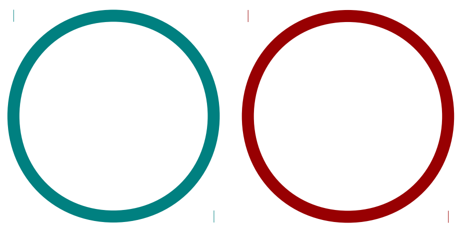
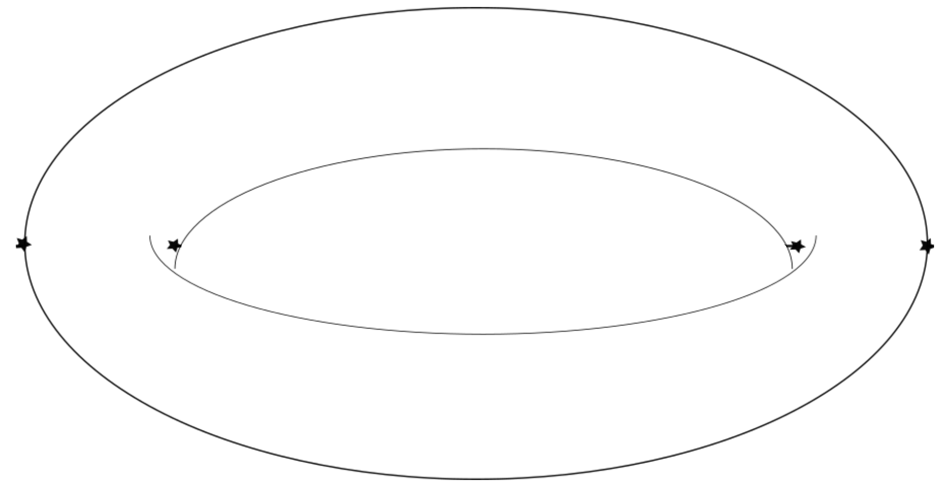
$f : \coprod_n$  $\hookrightarrow$ 

$$B_n[\Sigma/\mathbb{Z}_2] \curvearrowright \mathcal{Z}(f) : \mathcal{A} \boxtimes \mathcal{A} \cdots \boxtimes \mathcal{A} \to \mathcal{Z}[\mathbb{T}/\mathbb{Z}_2]$$

Summary

1. We interpreted the twisted reflection equation in Z_2-equivariant topology.

2. We showed any quantum symmetric pair defines a 2d Z_2-orbifold TQFT (assigning categories as invariants).

3. We obtained canonical orbifold braid group actions for surfaces with an involution (and isolated fixed points).

4. Hope: recover and generalise the $C^\vee C_n$ DAHA representations of D. Jordan and X. Ma.