Can $a = b$?

Tomasz Brzeziński

Swansea University (UK)
&
University of Białystok (Poland)

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Inspired by: Barry Mazur, *When is one thing equal to some other thing?* [in:] *Proof and other Dilemmas: Mathematics and Philosophy*, eds. B. Gold & R.A. Simons, MAA (2008)

Howbeit, for eaſie alteratiō of *equations*. I will propounde a fewe exāples, bicauſe the extraction of their rootes, maie the more aptly bee wroughte. And to auoide the tediouſe repetition of theſe woordes : is equalle to : I will ſette as I doe often in woorke vſe, a paire of paralleles, or Gemowe lines of one lengthe, thus: =======, bicauſe noe .2. thynges, can be moare equalle. And now marke theſe nombers.

Robert Recorde introducing $=$ in *The Whetstone of Witte* (1557)

The problem of meaning of =:

"=" means "the same"

But:

- Is 3 the same as 2.999999... or 3.0000...?
  Maybe better would be to say

  $$2.999999... \longrightarrow 3,$$

  i.e. 2.999999... *tends to be* the same as 3?
- Is 3 the same as $27/9$ or $134/67 + 1$?

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An ontological problem:

To judge whether the statement "*a* is the same as *b*" is true or false we need to (at least) believe that both *a* and *b* exist in the same realm.

- Do numbers exist, for example as specific sets?
- John von Neumann's model:

$$0 = \emptyset, \quad 1 = \{0\} = \{\emptyset\}, \quad 2 = \{0, 1\} = \{\emptyset, \{\emptyset\}\}, \quad 3 = \{0, 1, 2\}, \ldots$$

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Can really numbers be specific sets?

- If both models for numbers are correct then we would have:

$$\{\emptyset, \{\emptyset\}\} = 2 = \{\{\emptyset\}\}.$$

Does it look correct?

- Paul Benacerraf's story about Ernie and Johnny:
  - Boys educated individually by their parents.
  - Given basic ingredients (number 1, order $<$, successors) and rules they construct natural numbers are concrete sets, and derive their arithmetic properties ($+$, $\times$).
  - They are taught what numbers are useful for.
  - Their visions of numbers are very different.

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If taken literally as sets, depending on the model we would obtain different theorems about natural numbers, e.g.

Theorem (von Neumann & Ernie)

For any natural numbers, a, b, $a < b$ if and only if a is an element of b.

Theorem (Zermelo & Johnny)

For any natural numbers, a, b, a is an element of b if and only if $b = a + 1$.

If numbers are not sets then what are they?

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Penelope Maddy's set-theoretic realism

- Sets exist and can be experienced physically.

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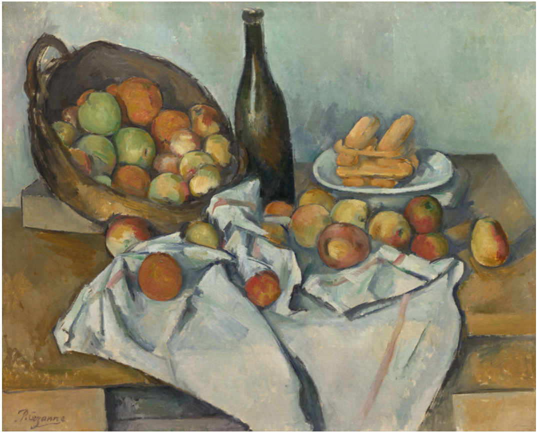

A Basket of Apples by Paul Cézanne (Arts Institute of Chicago)

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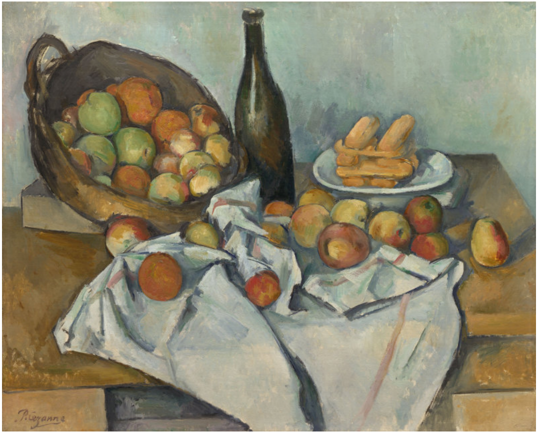

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- The structuralism approach (Paul Benaceraff, Charles Chihara, Michael Resnick, Stewart Shapiro):
  - Mathematical objects are *structures* or *patterns*.
  - For example every set which satisfies Peano's axioms is a pattern for natural numbers.
  - Individual numbers as *positions* in this pattern.

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Barry Mazur:

- First we need to "create" the realm or environment in which "sameness" or "=" can be precisely defined.
- The environment is provided by a suitable *category.*
- Within a category "sameness" means to be *isomorphic with respect to some universal property.*

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Categories

- objects: $\bullet$ (or $A, B, C, \ldots$);
- arrows (or morphisms): $\bullet \to \bullet$;
- composition $\circ$:

$$\begin{tikzcd} \bullet \arrow[r, "f"'] \arrow[rr, bend left, "g\circ f"] & \bullet \arrow[r, "g"'] & \bullet \end{tikzcd};$$

- identity arrow for each object, id_A for an object A;
- associative and identity laws:

$$h \circ (g \circ f) = (h \circ g) \circ f, \qquad \mathrm{id} \circ f = f = f \circ \mathrm{id}.$$

Examples of categories

- $\mathfrak{Set}$ – sets (objects), mappings (arrows).
- $\mathfrak{Vect}$ – vector spaces (objects), linear transformations (arrows).

All types of mathematical objects (sets, groups, rings, spaces, manifolds, graphs, complexes, etc.) can be gathered into categories specific to the type.

Isomorphisms, universal properties, the meaning of =

Two objects A and B are *isomorphic* if there exist arrows

$$\begin{tikzcd} A \arrow[r, shift right, "f"'] & B \arrow[l, shift right, "g"'] \end{tikzcd}, \qquad \text{such that} \qquad f \circ g = \mathrm{id}_B, \quad g \circ f = \mathrm{id}_A.$$

Main lesson from the category theory concerning the meaning of equality:

Rather than demanding that two objects within a given category be equal, one can demand they be *isomorphic* through a *particular* or *canonical* isomorphism, for example related to a specific *universal property*.

A property (P) is a *universal property* if (P) is satisfied by all objects in category $\mathfrak{C}$.

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An example of a universal property

An *initial object* is an object $*$, such that for any object $*$, $\bullet$, $\diamond$, etc., there is exactly one arrow from $*$ to this object:

$$
\begin{tikzcd}
 & * \\
* \arrow[ur, "\exists!"] \arrow[r, "\exists!"] \arrow[dr, "\exists!"] & \bullet \\
 & \diamond
\end{tikzcd}
$$

An example of a canonical isomorphism

Theorem

If both $*$ and $\diamond$ are initial objects in category $\mathfrak{C}$, then they are canonically isomorphic.

Proof. By definition there are unique arrows

$$* \xrightarrow{f} \diamond \qquad \text{and} \qquad \diamond \xrightarrow{g} * .$$

There composite are unique arrows:

$$* \xrightarrow{g \circ f} * \qquad \text{and} \qquad \diamond \xrightarrow{f \circ g} \diamond .$$

So they must be the unique arrows:

$$* \xrightarrow{\mathrm{id}_*} * \qquad \text{and} \qquad \diamond \xrightarrow{\mathrm{id}_\diamond} \diamond .$$

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An environment for natural numbers

The Peano category $\mathfrak{P}$:

- Objects are triples (X, b, s) consisting of:
  - a set X,
  - an element b of X called a *base point*,
  - a function

    $$s : X \to X,$$

    called the *successor map*.
- Arrows $(X, b, s) \longrightarrow (Y, c, t)$ are functions $f : X \to Y$ such that:
  - $f(b) = c$ (i.e. f preserves base points),
  - for all $x \in X$, $f(s(x)) = t(f(x))$ (i.e. f preserves the successor maps).

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Natural numbers as a structure in $\mathfrak{P}$

- Natural numbers are an object $(\mathbb{N}, b_{\mathbb{N}}, s_{\mathbb{N}})$ in $\mathfrak{P}$:
  - $\mathbb{N} = \{0, 1, 2, \ldots\}$,
  - $b_{\mathbb{N}} = 0$,
  - $s_{\mathbb{N}}(n) = n + 1$.

- $(\mathbb{N}, 0, s_{\mathbb{N}})$ is a *structure* and individual numbers n are *positions* in this structure determined by the number of applications of $s_{\mathbb{N}}$ to 0:

$$n = s_{\mathbb{N}}^{n}(0),$$

  e.g. $2 = s_{\mathbb{N}} \circ s_{\mathbb{N}}(0)$.

- $(\mathbb{N}, 0, s_{\mathbb{N}})$ is the initial object in $\mathfrak{P}$, since for any (X, b, s) we can construct the unique arrow

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Integers as a structure in $\mathfrak{P}$

- The integers are also an object $(\mathbb{Z}, 0, s_{\mathbb{Z}})$ in $\mathfrak{P}$

$$s_{\mathbb{Z}} : n \mapsto \begin{cases} |n|, & n < 0, \\ -n-1, & n \geq 0\,. \end{cases}$$

- The unique arrow $(\mathbb{N}, 0, s_{\mathbb{N}}) \to (\mathbb{Z}, 0, s_{\mathbb{Z}})$ comes out as:

$$f : n \mapsto \begin{cases} -\frac{n+1}{2}, & \text{if } n \text{ is odd}, \\ \frac{n}{2}, & \text{if } n \text{ is even}, \end{cases}$$

- This arrow has an inverse:

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$\mathbb{N} = \mathbb{Z}$ in $\mathfrak{P}$ but $2 \neq 2$

- The integers are canonically isomorphic to the natural numbers in $\mathfrak{P}$; they are *the same* in the Peano category, i.e.

$$(\mathbb{N}, 0, s_{\mathbb{N}}) = (\mathbb{Z}, 0, s_{\mathbb{Z}}).$$

- Note that 2 occupies the 'second' position in $\mathbb{N}$, as $2 = s_{\mathbb{N}}^2(0)$, while it occupies the fourth position in $\mathbb{Z}$,

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(the second position is occupied by -1). Hence:

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An environment that distinguishes $\mathbb{N}$ from $\mathbb{Z}$

- To distinguish between $\mathbb{N}$ and $\mathbb{Z}$ we need to view them as structures in a more elaborate category, e.g. the category $\mathfrak{S}$ of *semi-rings*.
- Objects are quintuples $(S, +_S, 0_S, \times_S, 1_S)$ satisfying a system of well-known axioms.
- Arrows

$$(S, +_S, 0_S, \times_S, 1_S) \to (T, +_T, 0_T, \times_T, 1_T),$$

are (structure preserving) functions $f : S \to T$ such that
  - $f(s +_S s') = f(s) +_T f(s'),$
  - $f(0_S) = 0_T,$
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An environment that distinguishes $\mathbb{N}$ from $\mathbb{Z}$

- The natural numbers together with addition and multiplication are an object in $\mathfrak{S}$, $(\mathbb{N}, +, 0, \times, 1)$.
- Also the integers with addition and multiplication are an object in $\mathfrak{S}$, $(\mathbb{Z}, +, 0, \times, 1)$.
- The natural numbers are the initial object in $\mathfrak{S}$.
- The canonical unique arrow

$$(\mathbb{N}, +, 0, \times, 1) \to (\mathbb{Z}, +, 0, \times, 1), \qquad n \mapsto n,$$

  is not onto, hence it is not an isomorphism.
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Some inconclusive conclusions

- The humble = sign creates a wealth of philosophical questions.
- One way to deal with the meaning of '=' is to view mathematical entities as structures and place each one of them in a suitable category.
- The equality of structures can only be discussed within the same environment or category.
- The categorical way of thinking instructs us that within a category the strength of 'the sameness' must be relaxed and '=' replaced by a canonical isomorphism (relative to some universal property).

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And finally... *The Adventure of Silver Blaze*

Gregory: “Is there any other point to which you would wish to draw my attention?”

Holmes: “To the curious incident of the public lecture on quantum homogeneous spaces.”

Gregory: “The lecturer didn’t talk about quantum homogenous spaces.”

Holmes: “That was the curious incident.”