Integrable structures arising from split symmetric pairs

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Goal of the talk

Construction of

1. eigenstates for quantum trigonometric spin Calogero-Moser systems,
2. eigenfunctions for boundary Knizhnik-Zamolodchikov-Bernard (KZB) operators,

in terms of vector-valued Harish-Chandra series.

Joint work with Kolya Reshetikhin.

Split symmetric pair $(\mathfrak{g}, \mathfrak{g}^\theta)$

1. Complex simple Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$, Killing form $(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$.

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2. Chevalley involution $\theta \in \mathrm{Aut}(\mathfrak{g})$,

$$\theta|_{\mathfrak{h}} = -\mathrm{Id}_{\mathfrak{h}}, \qquad \theta(e_\alpha) = -e_{-\alpha}$$

with $e_\alpha \in \mathfrak{g}_\alpha$ such that $(e_\alpha, e_{-\beta}) = \delta_{\alpha,\beta}$.

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Examples
$(\mathfrak{g}, \mathfrak{g}^\theta) = (\mathfrak{sl}_{\ell+1}(\mathbb{C}), \mathfrak{so}_{\ell+1}(\mathbb{C}))$ and $(\mathfrak{g}, \mathfrak{g}^\theta) = (\mathfrak{sp}_\ell(\mathbb{C}), \mathfrak{gl}_\ell(\mathbb{C}))$.
In both cases Chevalley involution $\theta(X) := -X^T$.

Vector-valued Harish-Chandra series

Notations $\{\alpha_1, \ldots, \alpha_\ell\}$ the base for Φ^+,

1. $\mathcal{R}$ the ring of rational trigonometric functions on $\mathfrak{h}$ generated by $\mathbb{C}[e^{-\alpha_1}, \ldots, e^{-\alpha_\ell}]$ and $(1 - e^{-2\alpha})^{-1}$ for $\alpha \in \Phi^+$.

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2. $\mathcal{R} \hookrightarrow \mathbb{C}[[e^{-\alpha_1}, \ldots, e^{-\alpha_\ell}]]$ (power series expansion in the sector $\mathfrak{h}_+ := \{h \in \mathfrak{h} \mid \Re(\alpha(h)) > 0 \quad \forall\, \alpha \in \Phi^+\}$).

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3. $\mathcal{D}_{\mathcal{R}}$: algebra of linear differential operators on $\mathfrak{h}$ with coefficients in $\mathcal{R}$.

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Definition

$$L := \sum_{k=1}^{\ell} \frac{\partial^2}{\partial x_k^2} + \sum_{\alpha \in \Phi^+} \left(\frac{1 + e^{-2\alpha}}{1 - e^{-2\alpha}} \right) \frac{\partial}{\partial h_\alpha} + 2 \sum_{\alpha \in \Phi^+} \frac{y_\alpha^2}{(e^\alpha - e^{-\alpha})^2} \in U(\mathfrak{g}^\theta) \otimes \mathcal{D}_{\mathcal{R}}$$

with

1. $\{x_1, \ldots, x_\ell\}$ *a linear basis of* $\mathfrak{h}$ *such that* $(x_i, x_j) = \delta_{i,j}$,
2. $h_\alpha \in \mathfrak{h}$ *such that* $(h, h_\alpha) = \alpha(h)$ *for all* $h \in \mathfrak{h}$.

Vector-valued Harish-Chandra series

Generic conditions on the spectral parameters:

$$\mathfrak{h}^*_{HC} := \{\lambda \in \mathfrak{h}^* \mid (2\lambda + 2\rho + \gamma, \gamma) \neq 0 \qquad \forall 0 \neq \gamma \in \mathbb{Z}_{\leq 0}\Phi^+\}$$

with $\rho := \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ and $\big(\cdot,\cdot\big) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ the dual of the non-degenerate bilinear form $\big(\cdot,\cdot\big)|_{\mathfrak{h}\times\mathfrak{h}}$ on $\mathfrak{h}$.

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Theorem

Let N be a finite dimensional $\mathfrak{g}^\theta$-module and give it the norm topology. Let $n \in N$ and $\lambda \in \mathfrak{h}^*_{HC}$. There exists a unique N-valued holomorphic function F^n_λ on $\mathfrak{h}_+$ of the form

$$F^n_\lambda(h) = \sum_{\gamma\in\mathbb{Z}_{\leq 0}\Phi^+} \Gamma^n_\gamma(\lambda) e^{(\lambda+\gamma)(h)}, \qquad \Gamma^n_\gamma(\lambda) \in N$$

satisfying $L(F^n_\lambda) = (\lambda + 2\rho, \lambda)F^n_\lambda$ and the initial condition $\Gamma^n_0(\lambda) = n$.

Vector-valued Harish-Chandra series

Terminology: F_λ^n is the N-valued Harish-Chandra series for $(\mathfrak{g}, \mathfrak{g}^\theta)$ with leading term $n \in N$ (for N the trivial $\mathfrak{g}^\theta$-representation, it is the usual Harish-Chandra series).

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Remarks

1. $L = L_\Omega$ is the $(\mathfrak{g}, \mathfrak{g}^\theta)$-radial component of the action of the Casimir element
$$\Omega := \sum_{k=1}^{\ell} x_k^2 + \sum_{\alpha \in \Phi} e_\alpha e_{-\alpha} \in Z(U(\mathfrak{g}))$$
acting by right-invariant differential operators on vector-valued spherical functions.

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2. Radial component map gives an algebra embedding $Z(U(\mathfrak{g})) \hookrightarrow U(\mathfrak{g}^\theta) \otimes \mathcal{D}_{\mathcal{R}}$, $C \mapsto L_C$ and
$$L_C(F^n_\lambda) = \xi_\lambda(C) F^n_\lambda, \qquad C \in Z(U(\mathfrak{g}))$$
with $\xi_\lambda : Z(U(\mathfrak{g})) \to \mathbb{C}$ the central character at λ.

Quantum trigonometric spin Calogero-Moser systems

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into a Hamiltonian by gauging with

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Quantum trigonometric spin Calogero-Moser Hamiltonian:

$$H := \delta L \delta^{-1} = \sum_{k=1}^{\ell} \frac{\partial^2}{\partial x_k^2} + \sum_{\alpha\in\Phi} \frac{1}{(e^{\alpha}-e^{-\alpha})^2}\left(\frac{(\alpha,\alpha)}{2} + y_\alpha^2\right) - (\rho,\rho)$$

with eigenfunction the gauged N-valued Harish-Chandra series

$$\mathbf{F}_\lambda^n := \delta F_\lambda^n.$$

Representation theoretic interpretation of F_λ^n

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1. Verma module $M_\lambda := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ with respect to Borel subalgebra $\mathfrak{b} := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ (**note:** M_λ is irreducible for $\lambda \in \mathfrak{h}^*_{HC}$).

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Lemma (Gelfand pair property)

Let $\lambda \in \mathfrak{h}^*_{HC}$.

1. $\mathrm{Dim}\left(\overline{M}_\lambda^{\mathfrak{g}^\theta}\right) = 1$.
2. *There exists a unique $v_\lambda = (v_\lambda[\mu])_{\mu\leq\lambda} \in \overline{M}_\lambda^{\mathfrak{g}^\theta}$ with $v_\lambda[\lambda] = m_\lambda$.*

Representation theoretic interpretation of F_λ^n

Theorem

Let N be a f.d. $\mathfrak{g}^\theta$-module, $\lambda \in \mathfrak{h}_{HC}^*$ and $\phi_\lambda \in \mathrm{Hom}_{\mathfrak{g}^\theta}(M_\lambda, N)$. Then

$$F_\lambda^{\phi_\lambda(m_\lambda)}(h) = \sum_{\mu \leq \lambda} \phi_\lambda(v_\lambda[\mu]) e^{\mu(h)}, \qquad h \in \mathfrak{h}_+.$$

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   Next step: the evaluation map is an isomorphism if N is a finite dimensional $\mathfrak{g}$-module and λ is sufficiently generic. In this case the vector-valued Harish-Chandra series are also eigenfunctions of boundary KZB operators.

KZB operators

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Setup

1. $\mathfrak{h}$-invariant element $r \in \mathcal{R} \otimes \mathfrak{g}^{\otimes 2}$.
2. $U(\mathfrak{g})^{\otimes s}$-valued differential operators on $\mathfrak{h}$:

$$D_i^{(s)} := \sum_{k=1}^{\ell} (x_k)_i \frac{\partial}{\partial x_k} - \sum_{j=1}^{i-1} r_{ji} + \sum_{j=i+1}^{s} r_{ij} \in U(\mathfrak{g})^{\otimes s} \otimes \mathcal{D}_{\mathcal{R}}$$

for $i = 1, \ldots, s$.

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Proposition

The following two statements are equivalent:

1. *For all $s \geq 2$ and $1 \leq i \neq j \leq s$,*

$$[D_i^{(s)}, D_j^{(s)}] = -\sum_{k=1}^{\ell} \frac{\partial r_{ij}}{\partial x_k} \Delta^{s-1}(x_k)$$

with $\Delta^{s-1} : U(\mathfrak{g}) \to U(\mathfrak{g})^{\otimes s}$ the $(s-1)^{th}$ iterated comultiplication.

2. *r is a solution of the classical dynamical Yang-Baxter equation (cdYBE).*

KZB operators

Classical dynamical Yang-Baxter equation (Gervais, Neveu, Felder):

$$\sum_{k=1}^{\ell}\Big((x_k)_3\frac{\partial r_{12}}{\partial x_k}-(x_k)_2\frac{\partial r_{13}}{\partial x_k}+(x_k)_1\frac{\partial r_{23}}{\partial x_k}\Big)$$
$$+[r_{12},r_{13}]+[r_{12},r_{23}]+[r_{13},r_{23}]=0$$

as identity in $\mathcal{R}\otimes\mathfrak{g}^{\otimes 3}$.

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Corollary

Write

$$\mathbf{V}:=V_1\otimes\cdots\otimes V_s$$

for finite dimensional $\mathfrak{g}$-modules $V_1,\ldots,V_s$. Let r be an $\mathfrak{h}$-invariant solution of the cdYBE. The associated differential operators $D_1^{(s)},\ldots,D_s^{(s)}$ pairwise commute when acting on $\mathbf{V}[0]$-valued functions on $\mathfrak{h}$.

KZB operators

Definition

The KZB operators are the differential operators $D_1^{(s)}, \ldots, D_s^{(s)}$ *associated to Felder's trigonometric solution*

$$r := -\sum_{k=1}^{\ell} x_k \otimes x_k - 2\sum_{\alpha\in\Phi} \frac{e_{-\alpha}\otimes e_{\alpha}}{1-e^{-2\alpha}}$$

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of the cdYBE.

Etingof, Schiffmann, Varchenko: common eigenfunctions of the KZB operators in terms of weight traces of products of vertex operators.

Vertex operators

Additional regularity assumptions:

$$\mathfrak{h}^*_{reg} := \{\lambda \in \mathfrak{h}^*_{HC} \mid \big(\lambda, \alpha^\vee\big) \notin \mathbb{Z} \quad \forall\, \alpha \in \Phi\}.$$

Proposition (Etingof, Varchenko)

Let V be a finite-dimensional $\mathfrak{g}$-module, μ a weight of V, and $\lambda \in \mathfrak{h}^*_{reg}$. We have a linear isomorphism

$$\mathrm{Hom}_{\mathfrak{g}}\big(M_\lambda, M_{\lambda-\mu} \otimes V\big) \xrightarrow{\sim} V[\mu]$$

mapping Ψ to $\big(m^*_{\lambda-\mu} \otimes \mathrm{Id}_V\big)(\Psi m_\lambda)$.

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For $v \in V[\mu]$ we write

$$\Psi^v_\lambda \in \mathrm{Hom}_{\mathfrak{g}}(M_\lambda, M_{\lambda-\mu} \otimes V)$$

for its preimage, called the **vertex operator** with leading term v.

Fusion

1. $\mathbf{V} = V_1 \otimes \cdots \otimes V_s$ with V_i finite dimensional $\mathfrak{g}$-modules.
2. $v_i \in V_i[\mu_i]$ for $i = 1, \ldots, s$, and $\mathbf{v} := v_1 \otimes \cdots \otimes v_s$.
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The fusion operator $J_{\mathbf{V}}(\lambda) : \mathbf{V} \xrightarrow{\sim} \mathbf{V}$ *is defined by*

$$J_{\mathbf{V}}(\lambda)\mathbf{v} := \big(m^*_{\lambda_0} \otimes \mathrm{Id}_{\mathbf{V}}\big)\big(\Psi^{v_1}_{\lambda_1} \otimes \mathrm{Id}_{V_2 \otimes \cdots \otimes V_s}\big) \cdots \big(\Psi^{v_{s-1}}_{\lambda_{s-1}} \otimes \mathrm{Id}_{V_s}\big)\Psi^{v_s}_{\lambda_s}(m_{\lambda_s}).$$

As identity in $\mathrm{Hom}_{\mathfrak{g}}\big(M_{\lambda_s}, M_{\lambda_0} \otimes \mathbf{V}\big)$,

$$\Psi^{J_{\mathbf{V}}(\lambda)\mathbf{v}}_{\lambda} = \big(\Psi^{v_1}_{\lambda_1} \otimes \mathrm{Id}_{V_2 \otimes \cdots \otimes V_s}\big) \cdots \big(\Psi^{v_{s-1}}_{\lambda_{s-1}} \otimes \mathrm{Id}_{V_s}\big)\Psi^{v_s}_{\lambda_s}.$$

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Remark (Etingof, Schiffmann). Common eigenfunctions of the KZB operators given by gauged versions of the trace functions $h \mapsto \mathrm{Tr}_{M_\lambda}\big(\Psi_{\lambda}^{J_{\mathbf{V}}(\lambda)\mathbf{v}} e^h\big)$ for $\mathbf{v} \in \mathbf{V}[0]$.

Boundary vertex operators

Lemma

$\lambda \in \mathfrak{h}^*$, *then*

1. $\mathrm{Dim}\left(M_\lambda^{*,\mathfrak{g}^\theta}\right) = 1$.
2. *There exists a unique* $f_\lambda \in M_\lambda^{*,\mathfrak{g}^\theta}$ *satisfying* $f_\lambda(m_\lambda) = 1$.

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$\lambda \in \mathfrak{h}^*$, then

1. $\mathrm{Dim}\big(M_\lambda^{*,\mathfrak{g}^\theta}\big) = 1$.
2. *There exists a unique $f_\lambda \in M_\lambda^{*,\mathfrak{g}^\theta}$ satisfying $f_\lambda(m_\lambda) = 1$.*

Consequence: If $\lambda \in \mathfrak{h}^*_{reg}$ and V is a finite dimensional $\mathfrak{g}$-module then

$$V \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{g}^\theta}\big(M_\lambda, V\big), \qquad v \mapsto \Psi_\lambda^{\flat,v}$$

with for $v \in V[\mu]$,

$$\Psi_\lambda^{\flat,v} := \big(f_{\lambda-\mu} \otimes \mathrm{Id}_V\big)\Psi_\lambda^v,$$

called the **boundary vertex operator** associated to v.

Boundary fusion operator

We have in $\mathrm{Hom}_{\mathfrak{g}^\theta}(M_{\lambda_s}, \mathbf{V})$:

$$\Psi_{\lambda}^{\flat, J_{\mathbf{V}}(\lambda)\mathbf{v}} = \left(\Psi_{\lambda_1}^{\flat, v_1} \otimes \mathrm{Id}_{V_2 \otimes \cdots \otimes V_s}\right)\left(\Psi_{\lambda_2}^{v_2} \otimes \mathrm{Id}_{V_3 \otimes \cdots \otimes V_s}\right) \cdots \Psi_{\lambda_s}^{v_s}$$

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Definition

For $\lambda \in \mathfrak{h}^*_{reg}$ the boundary fusion operator $J^\flat_{\mathbf{V}}(\lambda) : \mathbf{V} \xrightarrow{\sim} \mathbf{V}$ is defined by

$$J^\flat_{\mathbf{V}}(\lambda)\mathbf{v} := \Psi_\lambda^{\flat, J_{\mathbf{V}}(\lambda)\mathbf{v}}(m_\lambda).$$

Upshot: Fusion operator is the **highest to highest weight** component of products of vertex operators, while boundary fusion operator is the **highest weight to spherical** component of products of vertex operators:

$$J_{\mathbf{V}}(\lambda)\mathbf{v} := \left(m^*_{\lambda_0} \otimes \mathrm{Id}_{\mathbf{V}}\right)\left(\Psi_{\lambda_1}^{v_1} \otimes \mathrm{Id}_{V_2 \otimes \cdots \otimes V_s}\right) \cdots \left(\Psi_{\lambda_{s-1}}^{v_{s-1}} \otimes \mathrm{Id}_{V_s}\right)\Psi_{\lambda_s}^{v_s}(m_\lambda),$$

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Boundary KZB operators

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Integrable data

1. Folded r-matrices $r^{\pm} \in \mathcal{R} \otimes \mathfrak{g}^{\otimes 2}$:

$$r^{\pm} := \pm r + (\theta \otimes \mathrm{Id})(r)$$

with r Felder's trigonometric classical dynamical r-matrix.

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2. Folded k-matrix $\kappa \in \mathcal{R} \otimes U(\mathfrak{g})$ (with m multiplication map):
$$\kappa := m(\theta \otimes \mathrm{Id})(r) = \sum_{k=1}^{\ell} x_k^2 + 2 \sum_{\alpha \in \Phi} \frac{e_\alpha^2}{1 - e^{-2\alpha}}.$$

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Definition

The boundary KZB operators $D_i^{\flat,(s)} \in U(\mathfrak{g})^{\otimes s} \otimes \mathcal{D}_{\mathcal{R}}$ are

$$D_i^{\flat,(s)} := 2 \sum_{k=1}^{\ell} (x_k)_i \frac{\partial}{\partial x_k} - \sum_{j=1}^{i-1} r_{ji}^{+} - \sum_{j=i+1}^{s} r_{ij}^{-} - \kappa_i$$

for $i = 1, \ldots, s$.

Boundary KZB operators

Notations

1. $\mathbf{V} = V_1 \otimes \cdots \otimes V_s$ with V_i finite dimensional $\mathfrak{g}$-modules.
2. $v_i \in V_i[\mu_i]$ for $i = 1, \ldots, s$, and $\mathbf{v} := v_1 \otimes \cdots \otimes v_s$.
3. $\lambda_i := \lambda - \mu_s \cdots - \mu_{i+1}$ for $i = 0, \ldots, s$, with $\lambda_s := \lambda \in \mathfrak{h}^*_{reg}$.

Theorem

The gauged Harish-Chandra series $\mathbf{F}_\lambda^{J^\flat_{\mathbf{V}}(\lambda)\mathbf{v}} = \delta F_\lambda^{J^\flat_{\mathbf{V}}(\lambda)\mathbf{v}} : \mathfrak{h}_+ \to \mathbf{V}$ *satisfies*

$$D_i^{\flat,(s)}\big(\mathbf{F}_\lambda^{J^\flat_{\mathbf{V}}(\lambda)\mathbf{v}}\big) = \big((\lambda_i, \lambda_i + 2\rho) - (\lambda_{i-1}, \lambda_{i-1} + 2\rho)\big)\mathbf{F}_\lambda^{J^\flat_{\mathbf{V}}(\lambda)\mathbf{v}}$$

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for $i = 1, \ldots, s$.

Key tool: representation theoretic form of the particular vector-valued Harish-Chandra series as weighted **spherical to spherical** component of products of intertwiners:

$$F_\lambda^{J^\flat_{\mathbf{V}}(\lambda)\mathbf{v}} = \sum_{\mu \leq \lambda} \Big(\big(f_{\lambda_0} \otimes \mathrm{Id}_{\mathbf{V}}\big)\big(\Psi_{\lambda_1}^{v_1} \otimes \mathrm{Id}_{V_2 \otimes \cdots \otimes V_s}\big) \cdots \big(\Psi_{\lambda_{s-1}}^{v_{s-1}} \otimes \mathrm{Id}_{V_s}\big)\Psi_{\lambda_s}^{v_s}(v_\lambda[\mu])\Big) e^\mu.$$

Integrability

Recall the **quantum Hamiltonian** $H \in U(\mathfrak{g}^\theta) \otimes \mathcal{D}_{\mathcal{R}}$ given by

$$H := \sum_{k=1}^{\ell} \frac{\partial^2}{\partial x_k^2} + \sum_{\alpha \in \Phi} \frac{1}{(e^\alpha - e^{-\alpha})^2} \Big(\frac{(\alpha, \alpha)}{2} + y_\alpha^2 \Big) - (\rho, \rho)$$

and the **boundary KZB operators** $D_i^{\flat,(s)} \in U(\mathfrak{g})^{\otimes s} \otimes \mathcal{D}_{\mathcal{R}}$ given by

$$D_i^{\flat,(s)} := 2 \sum_{k=1}^{\ell} (x_k)_i \frac{\partial}{\partial x_k} - \sum_{j=1}^{i-1} r_{ji}^{+} - \sum_{j=i+1}^{s} r_{ij}^{-} - \kappa_i$$

for $i = 1, \ldots, s$ with folded r-matrices $r^{\pm}$ and a folded k-matrix κ.

Theorem

The differential operators $\Delta^{s-1}(H), D_1^{(s)}, \ldots, D_s^{(s)}$ pairwise commute in $U(\mathfrak{g})^{\otimes s} \otimes \mathcal{D}_{\mathcal{R}}$.

Integrability

Corollary

Folded r-matrices $r^{\pm} := \pm r + (\theta \otimes \mathrm{Id})(r)$ *satisfy mixed cdYBE*

$$2\sum_{k=1}^{\ell}\Big((x_k)_1\frac{\partial r_{23}^-}{\partial x_k} - (x_k)_2\frac{\partial r_{13}^-}{\partial x_k}\Big) = [r_{13}^-, r_{12}^+] + [r_{12}^-, r_{23}^-] + [r_{13}^-, r_{23}^-],$$

$$2\sum_{k=1}^{\ell}\Big((x_k)_1\frac{\partial r_{23}^+}{\partial x_k} - (x_k)_3\frac{\partial r_{12}^-}{\partial x_k}\Big) = [r_{12}^-, r_{13}^+] + [r_{12}^-, r_{23}^+] + [r_{13}^-, r_{23}^+],$$

$$2\sum_{k=1}^{\ell}\Big((x_k)_2\frac{\partial r_{13}^+}{\partial x_k} - (x_k)_3\frac{\partial r_{12}^+}{\partial x_k}\Big) = [r_{12}^+, r_{13}^+] + [r_{12}^+, r_{23}^+] + [r_{23}^-, r_{13}^+]$$

and $\kappa := m(\theta \otimes \mathrm{Id})(\theta)$ *the mixed classical dynamical reflection equation*

$$2\sum_{k=1}^{\ell}\Big((x_k)_2\frac{\partial(\kappa_1 + r^-)}{\partial x_k} - (x_k)_1\frac{\partial(\kappa_2 + r^+)}{\partial x_k}\Big) = [\kappa_1 + r^-, \kappa_2 + r^+].$$

Outlook

1. **Affine split symmetric pairs** related to
   1. quantum elliptic spin Calogero-Moser systems, including quantum Inozemtsev system (affine $\mathfrak{sl}_2$ case: Stefan Kolb).
   2. Boundary KZB equations involving Felder's elliptic solution of the classical dynamical Yang-Baxter equation with spectral parameter and an associated (folded) classical dynamical elliptic k-matrix with spectral parameter.
2. Quantum group versions involving **quantum symmetric spaces**: in progress!