Quantum groups acting on the nodal cubic

Manuel Martins and Ulrich Krähmer

Institute of Geometry, TU Dresden

Brief motivation

Classical notions		**Quantum analogues**
X set, G group	$\leadsto$	B algebra, A Hopf algebra,
$\alpha_x \colon G \to X$ surjective	$\leadsto$	$B \hookrightarrow A$ faithfully flat,
$\alpha \colon X \times G \to X$ action,	$\leadsto$	$\Delta \colon B \to B \otimes A$ right coideal
$H \leqslant G$ isotropy group	$\leadsto$	A/AB^+ quotient coalgebra

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Definition

An algebra embedding $B \subseteq A$ is **faithfully flat** when any chain complex

$$L \xrightarrow{f} M \xrightarrow{g} N\,, \quad g \circ f = 0$$

of B-modules is exact if and only if so is the induced complex of A-modules

$$A \otimes_B L \to A \otimes_B M \to A \otimes_B N.$$

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  is a quotient coalgebra. It is Hopf iff $AB^+ = B^+A$.
- The canonical map $\pi\colon A \to C$ induces a left coaction $\lambda\colon A \to C \otimes A$ (via Δ). Faithfully flatness implies that

$$B = A^{\mathrm{Co}\ C} := \{a \in A \mid \lambda(a) = \pi(1) \otimes a\}.$$

$B \subseteq A$ as a coalgebra Galois extension

- $B \subseteq A$ is a **Galois C-extension**, since the Galois map

$$\beta\colon A \otimes_B A \to C \otimes A, \quad a \otimes_B b \mapsto \pi(a_{(1)}) \otimes a_{(2)} b.$$

is bijective with inverse given by $\pi(a) \otimes b \mapsto a_1 \otimes_B S(a_2) b$.

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- Quantum: the most trivial bundle is $A = C \otimes B$, followed by $C \ltimes B$, followed by **cleft extensions**.

Definition

The extension $B \subseteq A$ is C-**cleft** if there exists a C-colinear map $\gamma\colon C \to A$ that is convolution invertible.

Cleftness and associated modules

Cleftness can be equivalently stated as $A \cong C \otimes B$ as left C-comodules right B-modules. Think $c \overset{\gamma}{\mapsto} c \otimes 1$ in $C \otimes B$ under the identification.

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Faithful flatness implies that the functor

$$-\Box_C A \colon \mathbf{Mod}^C \to \mathbf{Mod}^A_B$$

is an equivalence of categories. To finite-dimensional C-comodules V correspond finitely generated projective B-modules $\rightsquigarrow$ K-**theory of** B.

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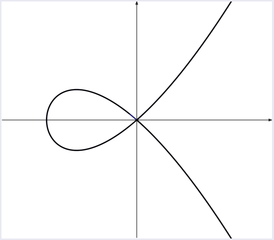
Question: can we generate $K_0(B)$ in this way?

C-cleft $\Rightarrow$ all associated modules are free since $V \Box_C C \otimes B \cong V \otimes B$.
$\Rightarrow$ answer is no, non-trivial elements in $K_0(B)$ are not reached.

The nodal cubic

Let k be a field.

The **nodal cubic** is plane curve in k^2 given by the equation $y^2 = x^2 + x^3$:



Nodal cubic over $\mathbb{R}$

Its coordinate ring is $B = k[x, y]/\langle y^2 - x^2 - x^3 \rangle$.

Singular curves cannot be homogeneous, right?

The nodal cubic as a quantum homogeneous space

Consider the algebra $\tilde{A}$ generated by x, y and invertible elements a, b satisfying

$$y^2 = x^2 + x^3, \quad a^3 = b^2,$$
$$ba = ab, \quad ya = ay, \quad bx = xb, \quad by = -yb,$$
$$xa^2 + axa + a^2x - a^2 + a^3 = 0, \qquad x^2a + xax + ax^2 + ax + xa = 0,$$

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From these relations, it follows that $\tilde{A}$ is a right free B-module (hence faithfully flat), with vector space basis

$$\left\{ a^i b^j (xa)^k x^m y^n \mid i \in \mathbb{Z}, k, m \in \mathbb{N}, j, n \in \{0, 1\} \right\}.$$

However, in this presentation, the commutation relations are too complicated for extensive computations.

The nodal cubic as a quantum homogeneous space

A Hopf algebra structure in $\tilde{A}$ is given by

$$\Delta(x) = 1 \otimes x + x \otimes a, \quad \Delta(y) = 1 \otimes y + y \otimes b,$$
$$\Delta(a) = a \otimes a, \qquad \Delta(b) = b \otimes b,$$
$$\varepsilon(x) = \varepsilon(y) = 0, \quad \varepsilon(a) = \varepsilon(b) = 1,$$
$$S(x) = -xa^{-1}, \quad S(y) = -yb^{-1}, \quad S(a) = a^{-1}, \quad S(b) = b^{-1},$$

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- $\tilde{A}$ is pointed but not connected. What else can we say about its properties, both as an algebra and as a coalgebra?
- Coproducts on powers of x and (xa) are also hard to compute, because of the commutation relations.

Playing around with the relations

Idea: find a smaller Hopf algebra which still admits a faithfully flat embedding of B as a right coideal subalgebra.

How: look for central and grouplike/primitive elements in $\tilde{A}$, generating (Hopf) ideals that intersect B trivially. E.g., $\langle a^3 - 1 \rangle$, since a^3 is grouplike and central.

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If the field contains a primitive 3rd root r of unity, then one can define

$$F := xa + (r+1)ax + \frac{r+2}{3}\left(a - a^2\right).$$

Think of F as a change of the variable (xa) that yields a nicer presentation.

Playing around with relations

Lemma

The class of F in $\tilde{A}/\langle a^3 - 1\rangle$ satisfies

$$aF = r^2 Fa, \qquad bF = Fb, \qquad yF = Fy,$$
$$xF = rFx + \frac{r+2}{3}aF + \frac{r-1}{3}F + \frac{1}{3}(a-1),$$

and

$$\Delta(F) = a \otimes F + F \otimes a^2.$$

Furthermore, the class of F^3 is central and primitive.

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Furthermore, the class of F^3 *is central and primitive.*

$\rightsquigarrow$ It is also "safe" to add the relation $F^3 = 0$.

The resulting quotient $A := \tilde{A}/\langle a^3 - 1, F^3\rangle$ is an iterated Ore extension and a finitely generated free $B-$module.

Main result - first half

Note that B is the subalgebra of A generated by x and y. Let C be the subalgebra generated by a, b and F.

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Theorem

- *$B \subseteq A$ is a quantum homogeneous space.*
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- *Multiplication in A defines an isomorphism $C \otimes B \cong A$ as left C-comodules and right B-modules, so A is a* **cleft extension** *of B.*

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- *Multiplication in A defines an isomorphism $C \otimes B \cong A$ as left C-comodules and right B-modules, so A is a* **cleft extension** *of B.*

The decomposition $A \cong C \otimes B$ follows explicitly from

$$\left\{ a^i b^j F^k x^m y^n \mid i, k \in \{0, 1, 2\}, m \in \mathbb{N}, j, n \in \{0, 1\} \right\}.$$

being a basis of A as a vector space. Since $B^+ = \langle x, y \rangle_B$, the projection $\pi \colon A \to A/AB^+$ restricts to an isomorphism $C \cong A/AB^+$.

Relation to $U_r(\mathfrak{sl}_2)$ and the small quantum group $u_r(\mathfrak{sl}_2)$

One can also define elements

$$E := xa - rax + \frac{1-r}{3}(a - a^2), \qquad K := a^2$$

which together with F satisfy the defining relations of $U_r(\mathfrak{sl}_2)$:

$$KE = r^2 EK, \quad KF = rFK, \quad [E, F] = \frac{K - K^2}{r - r^2}, \quad KK^{-1} = K^{-1}K = 1.$$

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Remark: Monomials in a, x and (xa) can be replaced by monomials in E, F and K (PBW-like basis), while b and y have the coproduct of the generators of Sweedler's infinite dimensional Hopf algebra H.
Indeed, A can be seen as a quotient of $U_r(\mathfrak{sl}_2) \otimes H$ by the relations

$$F^3 = 0, \quad K^3 = 1, \qquad y^2 = \frac{1}{27}E^3, \quad a^3 = b^2, \qquad b^2 = 1, \quad yb = -by$$

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The **small quantum group** $u_r(\mathfrak{sl}_2)$ is obtained by truncating $U_r(\mathfrak{sl}_2)$, i.e., imposing the relations

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The last two of these relations are already present in A, but the relation $E^3 = 0$ is equivalent to $y^2 = x^2 + x^3 = 0$. In this case, we obtain $u_r(\mathfrak{sl}_2) \otimes H_4$ as a quotient of A, where H_4 is Sweedler's 4-dimensional Hopf algebra.

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We have the Casimir element in A

$$\Omega := EF + \frac{r^2K + rK^2}{(r-r^2)^2} = (xa)^2 - a^2x - a^2x^2 + \frac{1}{3}$$

which is central.

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Proposition

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The Casimir element Ω has as minimal polynomial:

$$t^3 - \frac{1}{3}t + \frac{2}{27} = \left(t - \frac{1}{3}\right)^2 \left(t + \frac{2}{3}\right),$$

from where it follows that A is not semiprime.

Main theorem - second half

Theorem

If char $k \neq 2$ and $0 \neq I \subseteq A$ is a Hopf ideal, then $B \cap I \neq 0$. In other words, A is a minimal Hopf algebra containing B as a quantum homogeneous space.

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Proof: The group of group-likes of A is $\mathbb{Z}_3 \times \mathbb{Z}_2$. The presentation of A in terms of $U_r(\mathfrak{sl}_2)$ and H gives a complete characterization of the Yetter-Drinfel'd module of twisted primitives of A:

$$(1,1)\colon \langle x^2 + x^3 \rangle_k, \qquad (1,a)\colon \langle x, axa^2 \rangle_k, \qquad (1,b)\colon \langle y \rangle_k.$$

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A Hopf algebra map $A \to A/I$ such that $B \cap I = 0$ induces a injective map on the level of group-like elements and subsequently, on the level of the twisted primitives of A. This implies that $A \to A/I$ is injective, so $I = 0$ $\lightning$.

Food for thought

- Besides the Casimir, $y^2 = x^2 + x^3$ is also an element in the center. Is the whole center generated by these two?
- What is the nilradical of A?
- For other curves (as in Angela's work), are the corresponding extensions cleft as well?
- Can the K_0-group of B be obtained from A in some other way, such as a generalisation of a Mayer-Vietoris sequence to Hopf algebras or Hopf Galois extensions?