Homogeneous vector bundles over quantum spheres

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The topic

- *Quantization of vector bundles over a Poisson manifold is a* **natural next step** *after quantization of its function algebras*
- *Vector bundles are understood as* **projective modules** *over coordinate rings*
- *Presence of symmetry puts quantization problem in a context of* **representation theory**
- *Equivariant quantization is technically about* **complete reducibility** *of tensor product of representations*
- **Conravariant form** *is responsible for complete reducibility of tensor products*

One reason to quantize vector bundles:

- **Quantum stabilizer** *and its representations may be a problem. Amazingly it can be addressed through vector bundles.*

Classical vector bundles

A Lie group G, closed subgroup $K \subset G$, coset space $O = G/K$.

Function algebra $\mathbb{C}[O] \simeq \mathbb{C}[G]^K$

Vector bundle $E \to O$ with fiber $X \in K$-mod

Sections $O \to E$ form a projective $\mathbb{C}[O]$-module $\Gamma[O, X]$

Realization $\Gamma[O, X] = \mathrm{Hom}_K(\mathbb{C}, \mathbb{C}[G] \otimes X)$ (coinduced module)

In this talk:

$$G = SO(2n+1), \quad K = SO(2n), \quad O = \mathbb{S}^{2n}$$

a pseudo-Levi conjugacy class.

Quantization

$$G \dashrightarrow U_q(\mathfrak{g}), \quad \mathbb{C}[O] \dashrightarrow \mathbb{C}_q[O], \quad \Gamma[O, X] \dashrightarrow \Gamma_q[O, X], \quad K \dashrightarrow ?$$

Coideal subalgebra approach

$$t'_q = \left(\begin{array}{c|ccc|c} q^{-2n}-q^{-1} & & & & q^{-n-\frac{1}{2}} \\ \hline & -q^{-1} & & & \\ & & \ddots & & \\ & & & -q^{-1} & \\ \hline q^{-n-\frac{1}{2}} & & & & \end{array}\right)$$

is solution of Reflection Equation, $\lim_{q\to 1} t'_q = t' \in \mathbb{S}^{2n}$.

$\underline{t'_q \text{ defines:}}$

- embedding $\mathbb{C}_q[\mathbb{S}^{2n}] = \mathcal{A}_q \hookrightarrow U^* = U_q^*(\mathfrak{g})$
- a coideal subalgebra $\mathcal{B}_q \subset U_q(\mathfrak{g})$ s.t. $\mathcal{A}_q = \mathrm{Hom}_{\mathcal{B}_q}(\mathbb{C}, U^*)$.

Given a $\mathcal{B}_q$-module X,

is $\mathrm{Hom}_{\mathcal{B}_q}(\mathbb{C}, U^* \otimes X)$ a quantum vector bunlde ?

Representation theory of $\mathcal{B}_q$?

Operator quantization

Fix maximal torus $T \subset G$ of diagonal matrices and

$$t = \mathrm{diag}(-1, \dots, -1, \ 1\ , -1, \dots, -1) \in T \cap \mathbb{S}^{2n}$$

Polarization $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ and positive root set $\mathrm{R}^+_{\mathfrak{g}}$:

$$\varepsilon_i \pm \varepsilon_j, \quad i < j, \quad \varepsilon_i, \quad i, j = 1, \dots, n$$

Basis $\Pi_{\mathfrak{g}}$:

$$\alpha_1 = \varepsilon_1, \quad \alpha_2 = \varepsilon_2 - \varepsilon_1, \quad \dots, \quad \alpha_n = \varepsilon_n - \varepsilon_{n-1}$$

Basis of t-stabilizer $\Pi_{\mathfrak{k}}$:

$$\varepsilon_1 + \varepsilon_2, \quad \alpha_2, \quad \dots, \quad \alpha_n$$

Pseudo-Levi:

$$\Pi_{\mathfrak{k}} \not\subset \Pi_{\mathfrak{g}}$$

Operator realization ctd: base module M_λ

Define compound root vectors $f_{\varepsilon_i} \in U_q(\mathfrak{g})$, $i = 1, \ldots, n$, by

$$f_{\varepsilon_1} = f_{\alpha_1}, \quad f_{\varepsilon_i} = [f_{\varepsilon_{i-1}}, f_{\alpha_i}]_q = f_{\varepsilon_{i-1}} f_{\alpha_i} - q f_{\alpha_i} f_{\varepsilon_{i-1}}, \quad i > 1.$$

$U_q(\mathfrak{g})$-module M_λ of highest weight $\lambda \in \mathfrak{h}^*$, $q^{2(\lambda,\varepsilon_i)} = -q^{-1}$

$$\text{h.w.v} \quad 1_\lambda \in M_\lambda, \quad [f_{\alpha_1}, [f_{\alpha_1}, f_{\alpha_2}]_q]_{q^{-1}} 1_\lambda = 0 = f_{\alpha_i} 1_\lambda, \ i > 1.$$

$$M_\lambda = \mathrm{Span}\{f_{\varepsilon_1}^{m_1} \ldots f_{\varepsilon_n}^{m_n} 1_\lambda\}_{m_i \in \mathbb{Z}_+}$$

M_λ is irreducible and $\mathcal{A}_q \subset \mathrm{End}(M_\lambda)$

Projective equivariant $\mathcal{A}_q$-modules are candidates for QVB.

Proposition.

Let V be a finite dimensional $U_q(\mathfrak{g})$-module. Then all invariant idempotents from $\mathrm{End}(V \otimes M_\lambda)$ belong to $\mathrm{End}(V) \otimes \mathcal{A}_q$.

Problem reduces to complete reducibility of $V \otimes M_\lambda$.

Structure of $V \otimes M_\lambda$?

- What are highest weight submodules in $V \otimes M_\lambda$?
- When does $V \otimes M_\lambda$ split into direct sum of h.w. submodules?

If K were a Levi subgroup:

M_λ is a parabolic Verma module of h.w. λ.

- Highest weight submodules in $V \otimes M_\lambda$ are parabolically induced from irreducible $U_q(\mathfrak{k})$-submodules in V.
- Generically $V \otimes M_\lambda$ is a direct sum of h.w. submodules

Non-Levi case is special: no natural $U_q(\mathfrak{k})$ in $U_q(\mathfrak{g})$
no parabolic induction

Contravariant forms

We use shortcuts $U = U_q(\mathfrak{g})$, $U^{\pm} = U_q(\mathfrak{g}_{\pm})$

Define $\omega, \sigma \colon U \to U$

$$\sigma \colon e_\alpha \mapsto f_\alpha, \quad \sigma \colon h_\alpha \mapsto -h_\alpha, \quad \sigma \colon f_\alpha \mapsto e_\alpha,$$

$$\omega = \gamma^{-1} \circ \sigma, \quad \text{where} \quad \gamma = \text{antipode}$$

σ is algebra anti-automorphism and coalgebra map

A symmetric bilinear form $\langle ., . \rangle$ on a V-module is **contravariant** if

$$\langle xv, w \rangle = \langle v, \omega(x) w \rangle, \quad v, w \in V, \quad x \in U$$

Canonical contravariant form on $V \otimes M$

i) Every module of h.w. has a unique contravariant (Shapovalov) form, up to a scalar.

ii) The module is irreducible *iff* the Shapovalov form is non-degenerate.

- Let V, M be irreducible h.w. modules.
- Introduce canonical contravariant form on $V \otimes M$ as product of Shapovalov forms.
- Denote by $(V \otimes M)^{+} \subset V \otimes M$ the span of singular vectors.

Contravariant form and complete reducibility of $V \otimes M$

Theorem. (A.M.)

The following statements are equivalent:

1. $V \otimes M$ *is completely reducible*
2. *Canonical form is non-degenerate on* $(V \otimes M)^{+}$
3. *All h.w. submodules in* $V \otimes M$ *are irreducible*

Pseudo-parabolic modules

Let $\xi \in \mathfrak{h}^*$ be an integral dominant weight of $\mathfrak{k}$. Then

$$n_\alpha = (\xi, \alpha^\vee) + 1 \in \mathbb{N}, \quad q^{2(\xi+\lambda+\rho,\alpha)} = q^{n_\alpha(\alpha,\alpha)}, \quad \forall \alpha \in \Pi_\mathfrak{k}$$

In Verma module $\hat{M}_{\xi+\lambda}$ there are submodules $\hat{M}_{\lambda+\xi-n_\alpha\alpha}$, $\alpha \in \Pi_\mathfrak{k}$

<u>Definition:</u> Pseudo-parabolic module

$$M_{\xi,\lambda} = \hat{M}_{\xi+\lambda} / \sum_{\alpha \in \Pi_\mathfrak{k}} \hat{M}_{\lambda+\xi-n_\alpha\alpha}$$

Pseudo-parabolic category $\mathcal{O}_{\mathbb{S}^{2n}}$

Classical decomposition $V = \oplus_i X_i$ into sum of $\mathfrak{k}$-irreps of h.w. ξ_i.

Theorem.

For generic q:

1. $V \otimes M_\lambda \simeq \oplus_i M_{\xi_i,\lambda}$
2. *All* $M_{\xi_i,\lambda}$ *are irreducible.*

Definition: Pseudo-parabolic category $\mathcal{O}_{\mathbb{S}^{2n}_q}$ is a full subcat. in $\mathcal{O}$:

$$\mathrm{Ob}\,\mathcal{O}_{\mathbb{S}^{2n}_q} \subset \{\text{fin.dim. } U_q(\mathfrak{g})\text{-mod}\} \otimes M_\lambda,$$

- $\mathcal{O}_{\mathbb{S}^{2n}}$ is a module cat. over $U_q(\mathfrak{g})$-mod$^\circ$ (fin. dim.)
- $\mathcal{O}_{\mathbb{S}^{2n}}$ is semisimple Abelian
- $\mathcal{O}_{\mathbb{S}^{2n}}$ is isomorphic to $\mathfrak{k}$-mod$^\circ$

QVB over $\mathcal{A}_q$

Let $X \subset V$ be a $\mathfrak{k}$-submodule of h.w. ξ.

Let $P \in \mathrm{End}(V) \otimes \mathcal{A}_q$ be an idempotent,

$$P\colon V \otimes M_\lambda \to M_{\xi,\lambda}$$

Put $\Gamma_q[\mathbb{S}^{2n}, X] = P\big(\mathrm{End}(V) \otimes \mathcal{A}_q\big)$.

It is a left $U_q(\mathfrak{g})$-module and equivariant right $\mathcal{A}_q$-module

Proposition.

$\Gamma_q[\mathbb{S}^{2n}, X]$ *is an equivariant quantization of* $\Gamma[\mathbb{S}^{2n}, X]$.

Remark that $\Gamma_q[\mathbb{S}^{2n}, X]$ is a locally finite part of $\mathrm{Hom}_{\mathbb{C}}(M_\lambda, M_{\xi,\lambda})$.

Quantum symmetric pair and its representations

Get back to coideal subalgebra $\mathcal{B}_q \subset U_q(\mathfrak{g})$.

It is generated by entries of $\mathcal{R}_{12}(1 \otimes t'_q)\mathcal{R}_{21} \in U_q(\mathfrak{g}) \otimes \mathrm{End}(\mathbb{C}^{2n+1})$

Matrix t'_q defines a character $\chi_{t'_q} : \mathcal{A}_q \to \mathbb{C}$.

Let $\mathfrak{k}' \simeq \mathfrak{k}$ be the stabilizer of $t' = \lim_{q\to 1} t'_q$.

Theorem.

1. *Every finite dimensional $U_q(\mathfrak{g})$-module V is completely reducible over $\mathcal{B}_q$ for generic q.*
2. *Each irreducible $\mathcal{B}_q$-submodule is a deformation of a classical $U(\mathfrak{k}')$-submodule.*

An irreducible $\mathcal{B}_q$-submodule in V is the image of a $\mathcal{B}_q$-invariant projector $(\mathrm{id} \otimes \chi_{t'_q})(P) \in End(V)$, where

$$P \in End(V) \otimes \mathcal{A}_q$$

is a $U_q(\mathfrak{g})$-invariant indecomposable idempotent.

Bird's eye view

Additive categories (right $U_q(\mathfrak{g})$-mod setting)

1. finite-dimensional representations

   of quantum symmetric pair $U_q(\mathfrak{g}) \supset \mathcal{B}_q$

   $$\downarrow \mathrm{Hom}_{\mathcal{B}_q}(\mathbb{C}, U^* \otimes \{\cdot\}) \qquad \uparrow \mathbb{C}_\chi \otimes_{\mathcal{A}_q} \{\cdot\}$$

2. equivariant finitely generated $\mathcal{A}_q$-modules, $U_q(\mathfrak{g})$-loc. fin.

   $$\downarrow M_\lambda \otimes_{\mathcal{A}_q} \{\cdot\}, \qquad \uparrow \mathrm{Hom}^{\circ}_{\mathbb{C}}(M_\lambda, \{\cdot\})$$

3. pseudo-parabolic category $\mathcal{O}_{\mathbb{S}^{2n}}$

1. and 3. are equivalent semisimple Abelian

Parametrization of singular vectors in $V \otimes M$

Fix a pair of irreducible h.w. U-modules V, M.

- Regard ${}^*V$ and ${}^*M$ as U^+-modules.

Then

$$ {}^*M \simeq U^+/I_M^+, $$

where I_M^+ is left ideal in U^+.
Put

$$ V_M^+ = \mathrm{Hom}_{U_+}({}^*M, V) = \ker I_M^+ \subset V, $$

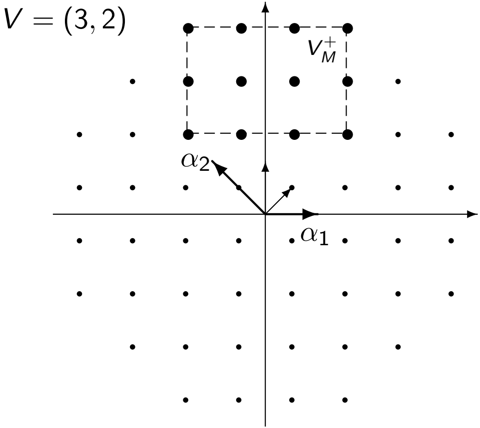
One has

$$ V_M^+ \simeq (V \otimes M)^+ \simeq M_V^+, $$

$$ V = V_M^+ \oplus \omega(I_M^+)V $$

Pull-back of the canonical form from $(V \otimes M)^+$ to V_M^+
"belongs" to dynamical Weyl group

Example: $\mathfrak{g} = \mathfrak{so}(5)$, $V = (3,2)$, $M = M_\lambda$



In general, for $\mathfrak{g} = \mathfrak{so}(2n+1)$ and $V = (\ell_1, \dots, \ell_n)$:

$$V_M^+ \simeq M_V^+ = \mathrm{Span}\{f_{\varepsilon_1}^{m_1} \dots f_{\varepsilon_n}^{m_n} 1_\lambda\}, \quad m_k = 0, \dots, \ell_k, \quad k = 1, \dots, n$$

Extremal twist

Consider the dual module ${}^*M$ of lowest weight and invariant form

$$M \otimes {}^*M \to \mathbb{C}.$$

$$\mathbb{C} \to {}^*M \otimes M \to U^+ \otimes U^-, \quad 1 \mapsto \mathcal{F}_M$$

Let $\gamma\colon U \to U$ be antipode.

Put $\Phi_M = \gamma^{-1}(\mathcal{F}_M^-)\mathcal{F}_M^+ \in U$ and define extremal twists $\theta_{V,M}$ by

$$\theta_{V,M}^+ \in \mathrm{End}(V_M^+)$$

Proposition.

The pull-back of canonical form under ismomorphism $V_M^+ \to (V \otimes M)^+$ *is*

$$\langle \theta_{V,M}(\,.\,),\,.\,\rangle$$