# Contractive idempotent functionals on locally compact quantum groups

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June 11, 2018

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- $B(G) \ni \omega \mapsto \lambda^u(\omega) \in \mathrm{C}_b(G)$ where $\lambda^u(\omega)(g) = \omega(U_g)$

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**Question**

What are the quantum counterparts of these results?

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- $L^\infty(\mathbb{G}), L^\infty(\widehat{\mathbb{G}}) \subset \mathsf{B}(L^2(\mathbb{G}))$, $\eta^\psi$ - the GNS map assigned to $\psi$

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- $\lambda^u$ is injective

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What are the $\lambda^u$ - images of

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- idempotent contractive functionals for QG's: Neufang, Salmi, Skalski, Spronk

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# Idempotent states and group-like projections, cont.

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# Idempotent states and group-like projections, cont.

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## Notation

Given an idempotent state $\omega \in \mathrm{C}_0^u(\widehat{\mathbb{G}})^*$ we write

$$P_\omega = \lambda^u(\omega) = (\omega \otimes \mathrm{id})(\mathbb{W})$$

# Contractive idempotent functionals, NSSS

## Polar decomposition of CIFs

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## Corollary

*Contractive idempotent functionals are preserved by $\widehat{\tau}^u$*

$$\omega = \omega \circ \widehat{\tau}_t^u \textit{ for all } t \in \mathbb{R}$$

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