A Counterexample for Subdifferential Valued Optimal Wealth

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Introduction

Recently, Deelstra-Pham-Touzi (2001) (DPT01) and Bouchard-Touzi-Zeghal (2004) (BTZ04) have worked on the problem without the usual smoothness assumption on the utility function. It is shown in (DPT01) that there exists an optimal terminal wealth X_* in $-\partial\tilde{U}(Y_*)$. When $\tilde{U}$ is differentiable, then $X_* = -\tilde{U}'(Y_*) = I(Y_*)$. When $\tilde{U}$ is not differentiable the set $\partial\tilde{U}$ is no longer a singleton and a smooth approximation technique (quadratic-inf-convolution) together with convergence analysis are used in (DPT01) to construct X_*.

A natural question is whether any random variable X_* chosen in $-\partial\tilde{U}(Y_*)$ is optimal for the utility maximization problem? It is conjectured in (DPT01) that the answer is positive. Here we give a simple counterexample to show that the answer is negative in general.

Model

We consider a finite time horizon T and frictionless market consisting of one bond, assumed constant, and one stock, a continuous semimartingale valued on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ satisfying the usual conditions. An agent starts with initial capital $x > 0$ and must decide at each time t how many units of the asset she wishes to hold in her self financing portfolio, we denote this by H_t. Her wealth process evolves as follows

$$X_t^{x,H} = x + \int_0^t H_u dS_u,$$

where the admissible H are predictable and S-integrable processes.

We consider only nonnegative terminal wealth random variables

$$\mathcal{X}(x) = \left\{ X \in L^0(\mathbb{R}^+, \mathcal{F}_T) : X = X_T^{x,H} \text{ for some admissible process } H \right\}.$$

Primal and Dual Problems

Assume $U : \mathbb{R} \to \overline{\mathbb{R}}$ such that $\mathrm{dom}(U) \subset \mathbb{R}^+$ is a utility function.

We can now formulate the utility maximization problem

$$V(x) := \sup_{X \in \mathcal{X}(x)} \mathbb{E}\big[U(X)\big].$$

Define the conjugate function

$$\tilde{U}(y) = \sup_{x>0} \big\{U(x) - xy\big\}.$$

We define the set of dual variables

$$\mathcal{Y}(y) := \left\{Y \in L^0(\mathbb{R}^+, \mathcal{F}_T) : \mathbb{E}[XY] \leq xy \text{ for all } X \in \mathcal{X}(x)\right\}.$$

We can now define the dual problem

$$W(x) := \inf_{y>0} \inf_{Y \in \mathcal{Y}(y)} \left(\mathbb{E}\left[\tilde{U}(Y)\right] + xy\right),$$

Assumptions

Assumption 1. $\mathcal{M} \neq \emptyset$, where $\mathcal{M}$ denote the set of equivalent local martingale measures for S. (absence of arbitrage in the market)

Assumption 2. $U(0) = 0$, $U(\infty) = \infty$ and U is concave and increasing on $\mathbb{R}^+$.

Then $\tilde{U}(0) = \infty$, $\tilde{U}(\infty) = 0$, convex, and decreasing. Define asymptotic elasticity for $\tilde{U}$ (see (DPT01)). (existence of a solution)

$$AE(\tilde{U}) := \limsup_{y \to 0} \frac{\delta_{-\partial \tilde{U}}(y)}{\tilde{U}(y)},$$

where $\delta_{-\partial \tilde{U}}(y) := \sup_{q \in -\partial \tilde{U}(y)}(qy)$ and $\partial \tilde{U}(y) := \left\{ \xi : \tilde{U}(z) \geq \tilde{U}(y) + \xi(z - y) \right\}$.

Assumption 3. $AE(\tilde{U}) < \infty$.

If U is C^1, strictly concave, and $U'(\infty) = 0$ then Assumption 3 is equivalent to $AE(U) = \limsup_{x \to \infty} \frac{xU'(x)}{U(x)} < 1$ (Kramkov-Schachermayer, 1999).

(DPT01) Result

Theorem 1. *Suppose that Assumptions 1, 2, 3 hold. Let $x > 0$ be such that $W(x) < \infty$, then*

1. *There exists some $y_* > 0$ and an optimal $Y_* \in \mathcal{Y}(y_*)$ such that,*

$$W(x) = \mathbb{E}\big[\tilde{U}\,(Y_*) + xy_*\big].$$

2. *There exists some X_* valued in $-\partial\tilde{U}\,(Y_*)$ such that*

$$X_* \in \mathcal{X}(x) \text{ and } V(x) = \mathbb{E}[U(X_*)].$$

3.

$$V(x) = W(x) = \inf_{y>0} \inf_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}\left[\tilde{U}\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right) + xy\right].$$

To deduce the last equality we have used Remark 3.9 from (BTZ04).

(DPT01) Conjecture

Conjecture 1. *For all random variables* X_* *valued in* $-\partial\tilde{U}\,(Y_*)$,

$$X_* \in \mathcal{X}(x) \text{ and } V(x) = \mathbb{E}\big[U(X_*)\big].$$

It is clearly true when the utility function is strictly concave since conjugate function $\tilde{U}$ is then differentiable and $\partial\tilde{U}(Y_*)$ reduces to a singleton. Thus the conjecture is nontrivial only when U is not strictly concave.

We first give a counterexample in a one period model and then by stopping a geometric Brownian motion we obtain the result in continuous time.

Discrete Time Counterexample

Consider a one-period model with stock price process $S = (S_0, S_1)$, where $S_0 = 1$ and S_1 is a discrete random variable taking the values $(2, \frac{1}{2})$ with probabilities

$$\mathbb{P}[S_1 = 2] = \frac{2}{3}, \quad \mathbb{P}\left[S_1 = \frac{1}{2}\right] = \frac{1}{3}.$$

In the notation of (DPT01) this corresponds to $d = 0$ and $\lambda = 0$, see Remark 2.1 therein for further discussion of this point. An elementary calculation shows that the (unique) risk-neutral probabilities are given by

$$\mathbb{Q}[S_1 = 2] = \frac{1}{3}, \quad \mathbb{Q}\left[S_1 = \frac{1}{2}\right] = \frac{2}{3}.$$

Moreover, the corresponding Radon-Nikodym density is given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}(\omega) = \begin{cases} \frac{1}{2} & S_1(\omega) = 2 \\ 2 & S_1(\omega) = \frac{1}{2} \end{cases}.$$

Assumption 1 is clearly satisfied as we have $\mathcal{M} = \{\mathbb{Q}\}$. In this framework we have an explicit description of the admissible terminal wealth set.

$$\mathcal{X}(x) = \big\{x + a(S_1 - 1) \, : \, a \in [-x, 2x]\big\}.$$

Define a utility function as follows

$$U(x) = \begin{cases} -\infty & x < 0 \\ \sqrt{x} & x \in [0, \frac{1}{16}) \\ \beta + 2(x-1) & x \in [\frac{1}{16}, 1) \\ \beta + \frac{1}{2}(x-1) & x \in [1, 2) \\ \log x + \gamma + 1 & x \in [2, \infty) \end{cases} . \tag{1}$$

The constants β and γ are given by $\frac{17}{8}$ and $\frac{13}{8} - \log 2$ respectively. They are chosen to ensure U is continuous on $\mathbb{R}^+$. It is easy to verify that, U satisfies Assumption 2, is not differentiable at $x = 1$ and is not strictly concave on the interval $\left[\frac{1}{16}, 2\right]$.

The conjugate function to U is given by

$$\tilde{U}(y) = \begin{cases} -\log y + \gamma & y \in (0, \frac{1}{2}) \\ \beta - y & y \in [\frac{1}{2}, 2) \\ \frac{1}{4y} & y \in [2, \infty) \end{cases} .$$

Observe that, for sufficiently small y, $\tilde{U}$ is differentiable and we may compute the asymptotic elasticity as

$$\mathrm{AE}(\tilde{U}) = \limsup_{y \to 0} \left(-\frac{1}{\log(y) + \gamma} \right) = 0.$$

This shows that Assumption 3 is satisfied.

The subdifferential of $\tilde{U}$ is given by

$$\partial\tilde{U}(y) = \begin{cases} -\frac{1}{y} & y \in (0, \frac{1}{2}) \\ [-2, -1] & y = \frac{1}{2} \\ -1 & y \in [\frac{1}{2}, 2) \\ [-1, -\frac{1}{16}] & y = 2 \\ -\frac{1}{4y^2} & y \in [2, \infty) \end{cases} .$$

For the remainder of the paper we fix $x = 1$. Now define the convex function

$$\begin{aligned} \varphi(y) \ &:= \ \mathbb{E}\left[\tilde{U}\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] + y \\ &= \ \frac{2}{3}\tilde{U}\left(\frac{y}{2}\right) + \frac{1}{3}\tilde{U}(2y) + y. \end{aligned}$$

We claim that

$$\inf_{y>0} \varphi(y) = \varphi(1),$$

or equivalently, $0 \in \partial\varphi(1)$. Using the addition and composition formulae for convex functions, we deduce

$$\partial\varphi(1) = \frac{1}{3}\partial\tilde{U}\left(\frac{1}{2}\right) + \frac{2}{3}\partial\tilde{U}(2) + 1.$$

Since $-1 \in \partial\tilde{U}\left(\frac{1}{2}\right) \cap \partial\tilde{U}(2)$ we have $0 \in \varphi(1)$.

The infimum in $W(1)$ is achieved at $y_* = 1$ and given that $\mathcal{M} = \{\mathbb{Q}\}$ the optimal dual variable is $Y_* = \frac{d\mathbb{Q}}{d\mathbb{P}}$. We may completely describe $-\partial\tilde{U}\left(Y_*\right)$ as

$$-\partial\tilde{U}\left(Y_*(\omega)\right) = \begin{cases} [1, 2] & S_1(\omega) = 2 \\ [\frac{1}{16}, 1] & S_1(\omega) = \frac{1}{2} \end{cases}.$$

Now choose X^+ as follows,

$$X^+(\omega) = \begin{cases} 2 & S_1(\omega) = 2 \\ 1 & S_1(\omega) = \frac{1}{2} \end{cases} \in -\partial\tilde{U}\left(Y_*(\omega)\right).$$

A simple calculation shows that

$$\mathbb{E}_{\mathbb{Q}}[X^+] = 2.\frac{1}{3} + 1.\frac{2}{3} = \frac{4}{3} > 1.$$

Since S is a martingale under $\mathbb{Q}$ and $S_0 = 1$ we have

$$\mathbb{E}_{\mathbb{Q}}[X] = \mathbb{E}_{\mathbb{Q}}\left[1 + a(S_1 - 1)\right] = 1, \quad \forall X \in \mathcal{X}(1).$$

It is now clear that $X^+ \notin \mathcal{X}(1)$, hence X^+ is not a feasible terminal wealth and therefore certainly not an optimal terminal wealth.

This shows that the conjecture is false in the discrete time case. Note that this does not affect the existence of an optimal solution, let

$$X^-(\omega) = \begin{cases} 1 & S_1(\omega) = 2 \\ \frac{1}{16} & S_1(\omega) = \frac{1}{2} \end{cases}$$

and choose a constant κ such that

$$\kappa \mathbb{E}_{\mathbb{Q}}[X^+] + (1 - \kappa)\mathbb{E}_{\mathbb{Q}}[X^-] = 1.$$

Now we have that the random variable $X_* := \kappa X^+ + (1 - \kappa)X^-$ satisfies $\mathbb{E}_{\mathbb{Q}}[X_*] = 1$ and is valued in $-\partial \tilde{U}\left(Y_*\right)$. These two conditions are sufficient to guarantee X_* is the unique solution to the primal problem.

Continuous Time Counterexample

We work on the same probability space as before, which supports a standard Brownian motion, denoted W_t. The construction of this example is similar to Example 5.1 in Kramkov and Schachermayer (1999). Define stopping time

$$\tau := \inf \left\{ t > 0 : \exp\left(W_t - \frac{1}{2}t \right) \notin \left(\frac{1}{2}, 2 \right) \right\}.$$

Clearly $\tau < \infty$ a.s and the process $Y_{t\wedge\tau} := \exp\left(W_{t\wedge\tau} - \frac{1}{2}(t\wedge\tau) \right)$ is bounded and hence a uniformly integrable martingale. A simple calculation reveals

$$\mathbb{P}[Y_\tau = 2] = \frac{1}{3}, \quad \mathbb{P}\left[Y_\tau = \frac{1}{2} \right] = \frac{2}{3}.$$

Define a time horizon $T = \tau$ and a security market made up of one stock with dynamics

$$S_t = \exp\left(-W_t + \frac{1}{2}t \right).$$

The market is complete and $(Y_t)_{0\leq t\leq T}$ is the density process of the risk-neutral measure, $\mathbb{Q}$. We define a utility function U as in Equation (1), having conjugate $\tilde{U}$. By the construction of the stopping time we have

$$\mathbb{E}[\tilde{U}(yY_T)] = \frac{2}{3}\tilde{U}\left(\frac{y}{2}\right) + \frac{1}{3}\tilde{U}(2y).$$

So that all the calculations go through identically as in the discrete case. Recalling that 1 is optimal, the inclusion $\mathcal{Y}(1) \supset \mathcal{M} = \{Y_T\}$ and the definition of $\mathcal{Y}(y)$ we deduce

$$\mathbb{E}_{\mathbb{Q}}[X] \leq 1 \tag{2}$$

for any $X \in \mathcal{X}(1)$. Again choose

$$X^+ = \begin{cases} 2 & Y_T(\omega) = 2 \\ 1 & Y_T(\omega) = \frac{1}{2} \end{cases} \in -\partial\tilde{U}\left(Y_*(\omega)\right).$$

As in discrete case we contradict Equation (2) so that $X^+ \notin \mathcal{X}(1)$. We now have a counterexample with continuous asset price process as claimed.