Connecting Singular and Switching Controls, with Applications

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Based on joint work with Pascal Tomecek at JP Morgan

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Outline:

- Motivation control problems
- Existing approaches and issues
- Main results
  - Connecting singular and switching controls
  - Linking switching controls and Dynkin's game
- Applications to reversible investment
  - Sufficient and necessary condition for smooth fit
  - Examples and discussions

1. Motivating problems

Reversible investment (Merhi and Zervos (2006))

$$\sup_{(\xi^+,\xi^-)} E[\int_0^\infty e^{-rt}\Pi(X_t, Y_t)dt - K^+ \int_0^\infty e^{-rt} d\xi_t^+ - K^- \int_0^\infty e^{-rt} d\xi_t^-]$$

- $Y_t = y + \xi_t^+ - \xi_t^-$, a firm's production capacity
- $X_t = \mu t + \sigma W_t$, e^{X_t} demand level or price of output commodity
- Π profit functions
- K^+, K^-, associated cost

Finite fuel/Bounded velocity/Monotone follower

- $\Pi(\cdot, \cdot) = H(W_t + Y_t)$, H concave
- $K^- = K^+ = 0$
- various constraints on the bound/rate of Y_t, or on the variation of Y_t

Bather and Chernoff (1967)
Benes, Shepp, Witsenhausen (1980)

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Risk management

- $\Pi(\cdot,\cdot) = H(W_t + Y_t)$, H concave
- $K^+ + K^- > 0$ (no arbitrage)
- Relax the constraint on Y_t
  Karatzas (1981)
  Harrison and Taksar (1983)

- Ma (1992) solves the problem with payoff $H(X_t)$ where

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + dY_t$$

(Ir)reversible investment

Davis, Dempster, Sethi and Vermes (1987)
Ma (1992)
Kobila (1993)
Boetius and Kohlmann (1998)
Oksendal (2000)
Scheinkman and Zariphopoulou (2001)
Chiarolla and Haussmann (2005)
Bank (2005)
Guo and Pham (2005)
Merhi and Zervos (2006)
Alveraz (2006)
Dixit and Pindyck (1994)

2. Existing approaches

- Assuming the value function to be a smooth solution to the Bellman PDE
- Finding a candidate for the value function by solving the Bellman equation
  - Assuming C^1 regularity at the boundary (ie “smooth fit”)
  - Dimension reduction technique
- Verification argument + the standard local time construction to show the optimality of the candidate solution

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Fundamental Issues

For most control problems, there is no regularity for either the value functions or the boundaries

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Viscosity solution approach

- Crandall and Lions (1983)
- Mathematical finance:
  Duffie and Zariphopoulou (1993)
  Shreve and Soner (1994)
  Duffie, Fleming, Soner and Zariphopoulou (1997)
- Mathematical economics:
  Guo and Pham (2005)
  Pham (2005)

Remaining issues (for higher dimension)

- Analytical aspect
  - Necessary and/or sufficient conditions for regularity properties
  - Characterization for the value function and the action regions when these regularity conditions fail
- Numerical aspect
  - "Right" class of functions (in terms of the degree of smoothness) for numerical approximation

3. Our approaches and results

- Building a new theoretical connection between general singular control and optimal switching problems
- Establishing regularity properties of the value function (and the smooth fit principle)
- Linking switching control and Dynkin's game

3.1. Connecting Singular and Switching Controls

- A singular control is a pair $(\xi_t^+, \xi_t^-)_{t\geq 0}$ of non-decreasing, adapted, left-continuous process with $\xi_0^+ = 0 = \xi_0^-$.
- (ξ^+, ξ^-) is admissible if $Y_t = y + \xi_t^+ - \xi_t^- \in (a, b)$ for all t and any given $y \in (a, b)$. Here $(a, b) \in R$ is an open interval.
- Admissible switching control $\alpha = (k_n, \tau_n)_{n\geq 0}$ consists of an increasing sequence of stopping times $(\tau_n)_{n\geq 0}$ such that $\tau_0 = 0$, $\tau_n \to \infty$, and an alternating sequence of new regime values $(\kappa_n)_{n\geq 0}$ that are assumed immediately after each of the stopping times. Here $\kappa_n \in \{0, 1\}$.

- There is one to one correspondence between switching control and its regime indicator function

$$I_t(\omega) = \sum_{n=0}^{\infty} \kappa_n 1_{\{\tau_n < t \leq t_{n+1}\}}$$

- A collection of switching controls $(\alpha(z))_{z\in(a,b)}$ is called consistent if
  - $I_0(z) = 1_{\{z\leq y\}}$ almost everywhere
  - $I_t(z)$ is increasing in z almost everywhere
  - The total variation of $I.(z)$ on $[0,t]$ is integrable over (a,b).

Bijection Theorem:

There is a 1-1 correspondence between a certain (i.e. consistent) class of switching controls and admissible singular control of finite variation.

$$I_t(z) := \lim_{s \uparrow t} 1_{\{Y_s > z\}}$$

$$Y_t = \int_y^b I_t(z)dz + \int_a^y (I_t(z) - 1))dz$$

with $Y_t = y + \xi_t^+ - \xi_t^- \in (a, b)$

Remarks:

- This connection between singular control and consistent class of switching control is generic.
- Previous techniques of connecting optimal stopping and singular control

  Bather and Chernoff (1967)
  Karatzas and Shreve (1984, 1985, 1986)
  El Karoui and Karatzas (1988, 1991)
  Baldursson and Karatzas (1997)
  Boetius (2001, 2005)

Representation Theorem:

Under proper conditions, solving singular control problem can be translated into solving switching control problems.

More precisely, consider

$$V(y) = \sup_{(\xi^+,\xi^-)} J(y, \xi^+, \xi^-)$$

with

$$J(y, \xi^+, \xi^-) = E\left[\int_0^\infty \Pi(t, Y_t)dt - \int_{[0,\infty)} \gamma_+(t)d\xi_t^+ - \int_{[0,\infty)} \gamma_-(t)d\xi_t^-\right]$$

and consider

$$m_+^*(z,k) := \sup_{\substack{\alpha\in\mathcal{B}\\ \kappa_0=k}} m_+(z,\alpha)$$

$$m_-^*(z,k) := \sup_{\substack{\alpha\in\mathcal{B}\\ \kappa_0=k}} m_-(z,\alpha)$$

with

$$\begin{aligned}
m_+(y,\alpha) &= E\left[\int_0^\infty \pi(t,y)I_t dt - \sum_{n=1}^\infty \gamma(\tau_n,\kappa_n)1_{\{\tau_n<\infty\}}\right] \\
m_-(y,\alpha) &= E\left[\int_0^\infty -\pi(t,y)(1-I_t)dt - \sum_{n=1}^\infty \gamma(\tau_n,\kappa_n)1_{\{\tau_n<\infty\}}\right] \\
\Pi(t,y_2) &- \Pi(t,y_1) = \int_{y_1}^{y_2} \pi(t,z)dz \\
I_t(z) &= \lim_{s\uparrow t} 1_{\{Y_s>z\}}
\end{aligned}$$

Then

$$V(y) - J(y,0,0) = \int_y^{\infty} m_+^*(z,0)dz + \int_{-\infty}^{y} m_-^*(z,1)dz$$

3.2. Sufficient Conditions for Regularity

Theorem: Assuming the existence of optimal admissible switching control, assuming certain integrability conditions and

$$\lim_{z \to y} E\left[\int_0^\infty |\pi(t,z) - \pi(t,y)|dt\right] = 0$$

Then V is C^1 and

$$V'(y) = m_+^*(y,1) - m_+^*(y,0)$$

In particular, since V is C^1 everywhere, leading to the smooth fit principle.
Key: Representation theorem + Lenhart and Belbas (1983), Pham (2005).

4. Dynkin's game

- A game of timing between two players, MAX and MIN
- For a fixed level z, MIN pays MAX at rate $\pi(t, z)$ while the game is in progress
- MAX and MIN each choose strategies on when to exit the game (the stopping times σ_+ and σ_- respectively)
- The player to exit first pays an amount to her opponent equal to $\gamma_+(\sigma_+)$ if MIN exits first, and $\gamma_-(\sigma_-)$ if MAX exits first.
- MAX chooses σ_+ to maximize her payoff, and MIN chooses σ_- in order to minimize MAX's payoff

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A Dynkin's game has a value if

$$\sup_{\sigma_+} \inf_{\sigma_-} E\left[D(\sigma_-, \sigma_+; z)\right] = \inf_{\sigma_-} \sup_{\sigma_+} E\left[D(\sigma_-, \sigma_+; z)\right].$$

with *the payoff of a Dynkin's game* given by

$$D(\sigma_-, \sigma_+; z) = \int_0^{\sigma_- \wedge \sigma_+} \pi(t, z)dt + \gamma_+(\sigma_+) 1_{\{\sigma_- \le \sigma_+\}} - \gamma_-(\sigma_-) 1_{\{\sigma_+ < \sigma_-\}}.$$

Value of Dynkin's game

Theorem: Under proper conditions, the value of the Dynkin's game exists, and

$$\begin{aligned} m_+^*(z,1) - m_+^*(z,0) &= \sup_{\sigma_+} \inf_{\sigma_-} E\left[D(\sigma_-,\sigma_+;z)\right] \\ &= \inf_{\sigma_-} \sup_{\sigma_+} E\left[D(\sigma_-,\sigma_+;z)\right]. \end{aligned}$$

5. Application to reversible investment

Problem

$$\sup_{(\xi^+,\xi^-)} E\left[\int_0^\infty e^{-\rho t} H(Y_t) X_t dt - \int_0^\infty e^{-\rho t} K^+ d\xi_t^+ - \int_0^\infty e^{-\rho t} K^- d\xi_t^-\right],$$

subject to

$$\begin{aligned}
Y_t &= y + \xi_t^+ - \xi_t^- \in \mathbb{R} \\
dX_t^x &= bX_t^x dt + \sqrt{2}\sigma X_t^x dW_t, \quad X_0 = x > 0 \\
H :& \ \text{concave, and continuous at } a \text{ and } b, \\
K^+ + K^- &> 0 \text{ (to prevent arbitrage).} \\
\mathcal{A}'_y &= \{(\xi^+, \xi^-) : \xi^\pm \text{ are left continuous, non-decreasing processes,} \\
& y + \xi_t^+ - \xi_t^- \in [a, b], \xi_0^\pm = 0; \\
& E\left[\int_0^\infty e^{-\rho t} d\xi_t^+ + \int_0^\infty e^{-\rho t} d\xi_t^-\right] < \infty.\left.\right\}.
\end{aligned}$$

5.1. Solution

- Solving the switching problem

$$v_k(x,z) := \sup_{\substack{\alpha \\ \kappa_0 = k}} E\left[\int_0^\infty e^{-\rho t}\left[h(z)X_t^x\right] I_t dt - \sum_{n=1}^\infty e^{-\rho \tau_n} K_{\kappa_n}\right]$$

- Proving consistency of switching controls
- Establishing the corresponding optimal singular control

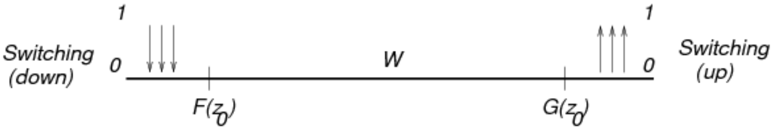
Key to explicit solutions

Value functions v_0 and v_1 for the optimal switching problem are the unique viscosity solutions with linear growth condition such that

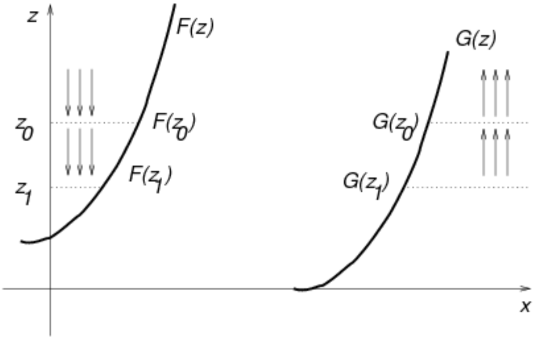
$$\begin{aligned} \min\left\{-\mathcal{L}v_0(x,z),\ v_0(x,z)-v_1(x,z)+K^+\right\} &= 0, \\ \min\left\{-\mathcal{L}v_1(x,z)-h(x,z),\ v_1(x,z)-v_0(x,z)+K^-\right\} &= 0, \end{aligned}$$

with $v_0(0^+,z)=0$ and $v_1(0^+,z)=\max\{-K^-,0\}$. Here $\mathcal{L}u(x,z)=\frac{1}{2}\sigma^2u_{xx}(x,z)+\mu u_x(x,z)-\rho u(x,z)$.

(1) For fixed z_0 switching control



(2) For general z, consistent switching controls

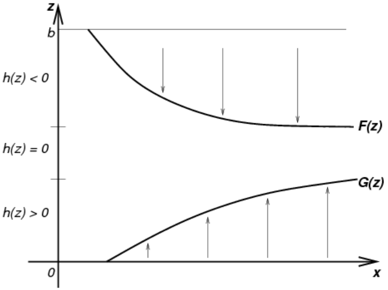


Explicit solution (continued)

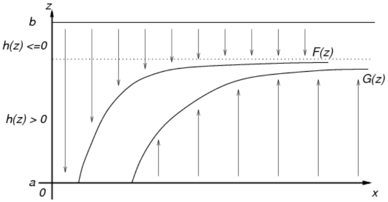
- The expansion region: $\{(x,z) : x \geq G(z)\}$
- The contraction region: $\{(x,z) : x \leq F(z)\}$
- The waiting region: $\{(x,z) : F(z) < x < G(z)\}$.

Two typical cases:

$K^- \geq 0$



$K^- < 0$



5.2. Necessary and Sufficient Conditions for Regularity

Theorem

- $V(x, y)$ is C^1 in x for all $(x, y) \in (0, \infty) \times [a, b]$.
- $V(x, y)$ is differentiable in y iff $(x, y) \in \mathcal{S}_0 \cup \mathcal{S}_1$ or H is differentiable at y.

Corollary When H is continuously differentiable, strictly increasing, strictly concave, with $H(0) = 0$, $H'(0^+) = \infty$, $H'(\infty) = 0$, boundaries between regions are indeed continuous and strictly increasing.

Remark When searching for a solution to

$$\max\{\frac{1}{2}\sigma^2 x^2 V_{xx}(x,y) + bxV_x(x,y) - rV(x,y) + \Pi(x,y),$$

$$V_y(x,y) - K^+, -V_y(x,y) - K^-\} = 0,$$

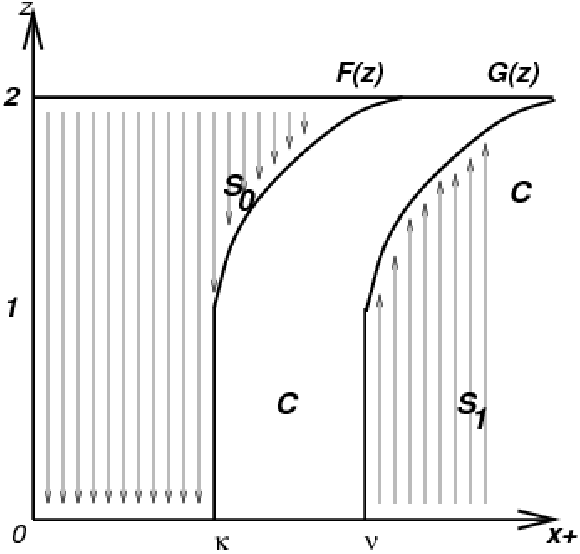
one would assume *a priori* smoothness for the value function and the boundary.
However, without the correct understanding of the regularity properties, one might end up searching in a wrong class of functions.

5.3. Examples

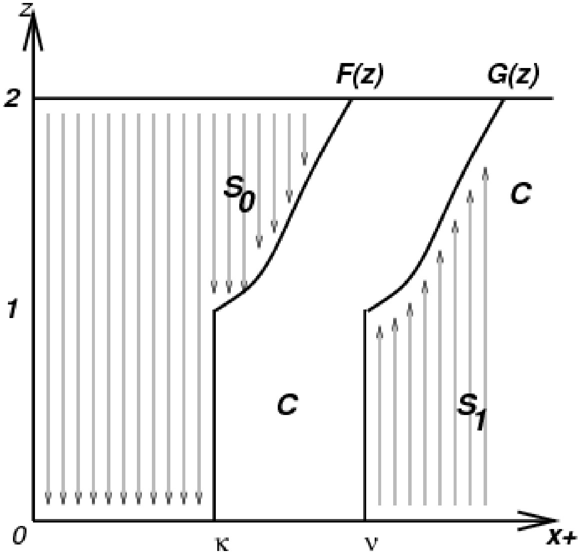
$$H(z) = \begin{cases} z, & z \leq 1, \\ \phi \frac{z^{\beta}-1}{\beta} + 1, & z > 1, \end{cases}$$

for some constant $\phi \in [0, 1]$. Here $(a, b) = (0, 2)$.

F, G not strictly increasing: H not strictly concave



F, G only C^0: H is not C^2



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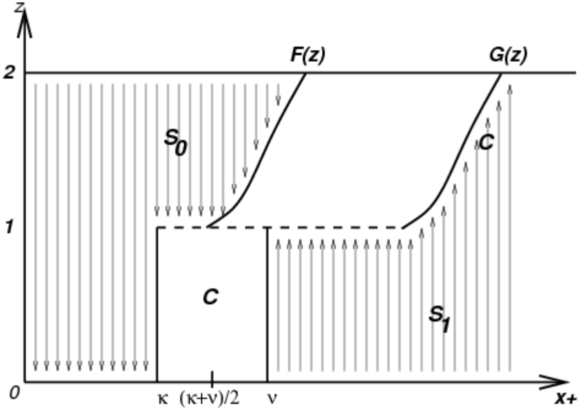
F, G NOT continuous: H is not C^1



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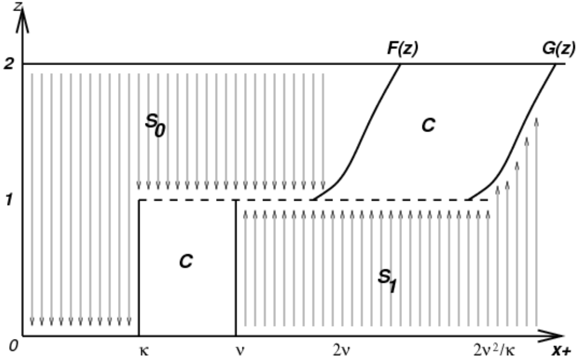
Interior of $\mathcal{C}$ NOT connected



6. Current work

- Control problem with state constraints

- Impulse controls and inventory controls

7. Reference

- X. Guo and P. Tomecek. "Connecting singular controls and switching controls." (SICON, to appear)
- X. Guo and P. Tomecek. "A class of constrained singular control problems and smooth fit principle." (Submitted)

THANK YOU VERY MUCH!