Mean-Variance Hedging in Stochastic Volatility Models Driven by Lévy Processes

Anke Wiese

Work in collaboration with

Jozef Kollar & Terence Chan

Heriot–Watt University, Edinburgh, UK

Further Developments in Quantitative Finance, ICMS
9th - 13th July 2007

Introduction

Aim: Option Hedging in Stochastic Volatility Models

$$\begin{aligned} \mathrm{d}S_t &= S_{t-}\big\{b(t, V_{t-})\,\mathrm{d}t + \sigma(t, V_{t-})\,\mathrm{d}B_t + \delta(t, V_{t-})\,\mathrm{d}J_t\big\} \\ \mathrm{d}V_t &= g(t, V_{t-})\,\mathrm{d}t + \gamma(t, V_{t-})\,\mathrm{d}L_t. \end{aligned}$$

B	standard Wiener process
J	pure jump Lévy process
L	Lévy process

Here:

- B, J, L independent

Continuous Models (Uncorrelated Version)

Hobson 2004

$$
\begin{aligned}
\mathrm{d}S_t &= S_t\{b(t, V_t)\,\mathrm{d}t + \sigma(t, V_t)\,\mathrm{d}B_t\} \\
\mathrm{d}V_t &= g(t, V_t)\,\mathrm{d}t + \gamma(t, V_t)\,\mathrm{d}W_t,
\end{aligned}
$$

where W Wiener process.

In particular: Heston 1993

$$
\begin{aligned}
\mathrm{d}S_t &= S_t\{b(V_t)\,\mathrm{d}t + \sqrt{V_t}\,\mathrm{d}B_t\} \\
\mathrm{d}V_t &= \kappa(m - V_t)\,\mathrm{d}t + \gamma\sqrt{V_t}\,\mathrm{d}W_t.
\end{aligned}
$$

Lévy Models (Uncorrelated Version)

Benth & Meyer-Brandis 2005, Barndorff-Nielsen & Shephard 2001

$$
\begin{aligned}
\mathrm{d}S_t &= S_t\big\{(\mu + bV_t)\,\mathrm{d}t + \sqrt{V_t}\,\mathrm{d}B_t\big\} \\
\mathrm{d}V_t &= -\lambda V_t\,\mathrm{d}t + \mathrm{d}L_{\lambda t},
\end{aligned}
$$

where L pure jump subordinator.

Li, Wells & Yu 2006

$$
\begin{aligned}
\mathrm{d}S_t &= S_{t-}\big\{b\,\mathrm{d}t + \sqrt{V_{t-}}\,\mathrm{d}B_t + \mathrm{d}J_t\big\} \\
\mathrm{d}V_t &= \kappa(m - V_t)\,\mathrm{d}t + \gamma\sqrt{V_{t-}}\,\mathrm{d}W_t + \mathrm{d}L_t,
\end{aligned}
$$

where W Wiener process and J and L pure jump Lévy processes.

Important Ideas

- Unifying approach for this class of models
- Tractability
- Signed variance-minimal martingale measure can be explicitly determined
- Positivity of variance-minimal martingale measure does not need to be assumed, applies also in a more general semimartingale framework. (Arai 2005, Lim 2005)
- Hedging measure and variance-minimal martingale measure do not coincide
- Derivation 3-dimensional PIDE
- Mean-variance hedging strategy can be numerically determined.

Mean-Variance Hedging Problem: Projection Problem

$H \in L^2(P)$ contingent claim, $T < \infty$ time horizon.

Objective: Find solution $\theta^\star$ of optimization problem

$$\min \left\{ \| H - \int_0^T \theta_u \, \mathrm{d}S_u \|_2 : \theta \in \Theta \right\}.$$

$$\Theta = \Theta_{CK} = \Big\{ \theta \in L(S) : \int_0^T \theta_u \mathrm{d}S_u = L^2 - \lim \int_0^T \theta_u^n \mathrm{d}S_u$$
$$\text{and } \int_0^t \theta_u \mathrm{d}S_u = \lim \int_0^t \theta_u^n \mathrm{d}S_u \text{ in probability } \forall t$$
$$\text{for a sequence } \theta^n \in L^2(S) \Big\}.$$

(Delbaen & Schachermayer 1996, Schweizer 1997, Gourieroux, Laurent & Pham 1997, Rheinländer & Schweizer 1997, W. 1998, Arai 2005, Lim 2005, Cerny & Kallsen 2005)

The Signed Variance-Optimal Martingale Measure

Problem: Find the (signed) local martingale measure $\tilde{P}$ that minimizes

$$\left\| \frac{\mathrm{d}Q}{\mathrm{d}P} \right\|_{L^2(P)}$$

over all signed local martingale measures Q with square-integrable density.

Characterization of the *variance-optimal martingale measure*: projection of $H \equiv 1$ on the orthogonal complement of $(\Theta \cdot S)_T$ (normalised by the expected value of this projection).
(Schweizer 1996, Delbaen & Schachermayer 1996)

The Optimal Hedge for a Constant Liability

Assume that

$$\int_0^T \frac{\left(\delta(u, V_{u-})E(J_1) + b(u, V_{u-})\right)^2}{\left(\sigma(u, V_{u-})^2 + \delta(u, V_{u-})^2 \int_{\mathbb{R}} x^2 \nu(dx)\right)} \, \mathrm{d}u \in L^1(P),$$

where ν denotes the Lévy measure of J.
Define

$$\begin{aligned} Z_t =& 1 - \int_0^t \lambda_u Z_{u-} \, \mathrm{d}S_u = \mathcal{E}(-\int \lambda_u \, \mathrm{d}S_u)_t, \\ \theta_t =& \lambda_t Z_{t-} = \lambda_t (1 - \int_0^{t-} \theta_u \, \mathrm{d}S_u). \end{aligned}$$

Then θ is the variance-optimal hedging strategy for hedging the constant $H \equiv 1$. Here

$$\lambda_t = \frac{1}{S_{t-}} \; \frac{\delta(t, V_{t-})E(J_1) + b(t, V_{t-})}{\left(\sigma(t, V_{t-})^2 + \delta(t, V_{t-})^2 \int_{\mathbb{R}} x^2 \nu(dx)\right)} = \frac{dA_t}{d\langle M \rangle_t},$$

where A and M are given by the Doob-Meyer decomposition of S.

Idea of Proof

Essential feature of model:

can find a localizing sequence of stopping times $(\tau_n)_n$ such that

$$\lambda Z_{-}^{\tau_n} 1_{(0,\,\tau_n]}$$

is the variance-optimal hedging strategy for hedging 1 unit with the stopped process S^{τ_n}. Note that due to projection argument

$$\left\| \mathcal{E}\left(\int -\lambda 1_{(0,\tau_n]} \, \mathrm{d}S \right)_T \right\|_2 \leq 1.$$

The Signed Variance-Optimal Martingale Measure

Corollary:
Assume $E[Z_T] \neq 0$. Then the signed measure $\tilde{P}$ defined by

$$\frac{\mathrm{d}\tilde{P}}{\mathrm{d}P} = \frac{Z_T}{E[Z_T]} = \frac{\mathcal{E}\left(-\int \lambda_t \,\mathrm{d}S_t\right)_T}{E[\mathcal{E}\left(-\int \lambda_t \,\mathrm{d}S_t\right)_T]}$$

is the signed variance-minimal martingale measure. That is the density of $\tilde{P}$ with respect to P has minimal L^2-norm among all (signed) local martingale measures with square-integrable density.

Remarks:

- No Markovian structure needed here.
- $E[Z_T] = 0 \Leftrightarrow Z_T = 0 \Leftrightarrow 1 = \int_0^T \theta_t^1 \,\mathrm{d}S_t$ P-a.s. If this is the case, the variance-optimal martingale measure does not exist (Schweizer 1996, Schachermayer).

Positivity of the Variance-Minimal Martingale Measure

Note that

$$Z_T = \exp\Big(- \int_0^T \lambda_t \, \mathrm{d}S_t - \int_0^T \lambda_t^2 \, \mathrm{d}[S, S]_t^c \Big) \times \prod_{t \le T} \left\{ 1 - \frac{(\delta(t, V_{t-}) E(J_1) + b(t, V_{t-}))\delta(t, V_{t-})}{\sigma(t, V_{t-})^2 + \delta(t, V_{t-})^2 \int_{\mathbb{R}} x^2 \nu(dx)} \Delta J_t \right\} \times \exp\left\{ + \lambda_t \delta(t, V_{t-}) \Delta J_t \right\}.$$

Thus

$$Z_T > 0 \quad \Leftrightarrow \quad \frac{\delta(\cdot, V_-) E(J_1) + b(\cdot, V_-)}{\sigma(\cdot, V_-)^2 + \delta(\cdot, V_-)^2 \int_{\mathbb{R}} x^2 \nu(dx)} \delta(\cdot, V_-) \Delta J < 1$$

$$\{Z_T = 0\} \Leftrightarrow \Big\{ \text{there exists a } t \le T \text{ such that } \frac{\delta(t, V_{t-}) E(J_1) + b(t, V_{t-})}{\sigma(t, V_{t-})^2 + \delta(t, V_{t-})^2 \int_{\mathbb{R}} x^2 \nu(dx)} \delta(t, V_{t-}) \Delta J_t = 1 \Big\}$$

Market Extension and the Hedging Measure

Gourieroux, Laurent & Pham 1997, Rheinländer and Schweizer 1997, Arai 2005 (underlying framework: S locally bounded and $Z_T > 0$)

Market extension $Y = (1/Z, S/Z)$ and new probability measure Q

$$\frac{\mathrm{d}Q}{\mathrm{d}\tilde{P}} = Z_T / E_{\tilde{P}}[Z_T].$$

Hedging problem is equivalent to

$$\text{minimize} \quad \left\| \frac{H}{Z_T} - \int_0^T \psi_s dY_s \right\|_{L^2(Q)} \qquad \text{over all } \psi \in L^2(Y, Q).$$

The Signed Case

Assume that there is an equivalent martingale measure with square-integrable density. Assume further that $P\{Z_T = 0\} = 0$. Then the probability measure Q by

$$\frac{\mathrm{d}Q}{\mathrm{d}P} = Z_T^2 / E[Z_T^2]$$

is well-defined, and

$$\frac{1}{Z_T}\{\int_0^T \theta_t \, dS_t : \theta \in \Theta\} = \{\int_0^T \psi_t \, \mathrm{d}Y_t : \psi \in L^2(Y)\}.$$

Remark: Valid more generally for locally square-integrable semimartingales (in $\mathcal{H}^2_{loc}$).

Measure Transform

Representation theorem for absolutely continuous measures $\bar{P}$ (Chan 1999):

$$\frac{\mathrm{d}\bar{P}}{\mathrm{d}P}\Big|_{\mathcal{F}_t} = \mathcal{E}\left(\int G_u dB_u + \iint_{\mathbb{R}} h(s,x) N(ds,dx) + \int F_u dB_u^V + \iint_{\mathbb{R}} f(s,x) N^V(ds,dx) \right)_t,$$

for processes G, F and $h(t,x)$, $f(t,x)$. Moreover, the processes G and F determine the $\bar{P}$-Wiener processes and h and f the compensators of the pure jump components.

Dynamics Under Q

Under Q, $Y = (1/Z, S/Z)$ is given by

$$\mathrm{d}\left(\frac{1}{Z}\right)_t = \left(\frac{1}{Z}\right)_{t-}\left\{\sigma\lambda S_- dB_t^Q + \int_{\mathbb{R}} h^0(u,x) N^Q(du,dx)\right\}$$
$$\mathrm{d}\left(\frac{S}{Z}\right)_t = \left(\frac{S}{Z}\right)_{t-}\left\{\sigma(1+\lambda S_{t-}) dB_t^Q + \int_{\mathbb{R}} h^1(u,x) N^Q(du,dx)\right\}$$

where B^Q is a Q-Wiener process, the compensator measure of J is given by $(-2\delta\lambda S_- x + 1)\,\nu(dx)du$, and

$$h^0(t,x) = -\frac{\delta\lambda S_- x}{1-\delta\lambda S_- x} = h^0(t, V_{t-}, x),$$
$$h^1(t,x) = \frac{\delta x(1-\lambda S_-)}{1-\delta\lambda S_- x} = h^1(t, V_{t-}, x).$$

Variable transforms all given directly by model parameters.

The Volatility Dynamics

The volatility process satisfies

$$dV_t = \tilde{g}(t, V_{t-})dt + \gamma(t, V_{t-})d\tilde{L}_t$$

where

$$\tilde{g}(t, V_{t-}) = g(t, V_{t-}) + \gamma(t, V_{t-})\big(a^V + F_t + \int_{\mathbb{R}} xf(t,x)\nu^V(dx)\big),$$

and F and f are determined by the martingale representation of

$$\exp\left\{-\int_0^T \frac{\big(\delta(t, V_{t-})E(J_1) + b(t, V_{t-})\big)^2}{\big(\sigma(t, V_{t-})^2 + \delta(t, V_{t-})^2 \int_{\mathbb{R}} x^2\nu(dx)\big)} \mathrm{d}t\right\}$$

with respect to the Lévy process L (driving the volatility process V).

Features of Measure Transform

- Measure transform preserves Markov property.
- Lévy property is not preserved (Esche & Schweizer 2005).
- Restrict to hedging strategies that only depend on S and V.

Numerical Method: Associated PIDE

$$u(t, 1/Z_t, (S/Z)_t, V_t) := E_Q[H/Z_T \big| \{1/Z_u, (S/Z)_u, V_u : u \le t\}]$$
$$= E_Q[G(S_T/Z_T, 1/Z_T, V_T) \big| \{1/Z_u, (S/Z)_u, V_u : u \le t\}]$$

Associated PIDE for $u = u(t, x, y, v)$

$$\frac{\partial u}{\partial t} + \tilde{g}(t,v)\frac{\partial u}{\partial v} + \frac{1}{2}\left(\gamma^2 \frac{\partial^2 u}{\partial v^2} + (y\sigma(1+\lambda S_-))^2 \frac{\partial^2 u}{\partial y^2}\right.$$
$$\left. + x^2(\sigma\lambda S_-)^2 \frac{\partial^2 u}{\partial x^2}\right) + xy\sigma\lambda S_- \sigma(1+\lambda S_-)\frac{\partial^2 u}{\partial x \partial y}$$
$$+ \int [u(t, x + xh^0(t,z), y + yh^1(t,x), v) - u(t,x,y,v)$$
$$- xh^0(t,z)\frac{\partial u}{\partial x} - yh^1(t,z)\frac{\partial u}{\partial y}]\nu_t^Q(dz)$$
$$+ \int [u(t,x,y,v+\gamma z) - u(t,x,y,v) - \gamma z \frac{\partial u}{\partial v}]\nu_t^{V,Q}(dz) = 0.$$

Hedging

Variance-optimal hedging strategy given by

$$\theta^* := \psi^1 - \lambda Z_- \left\{ \int_0^{\cdot -} \psi dY - \psi^{tr} Y_- \right\}$$

where $Y = (1/Z, S/Z)$ and $\psi = (\psi^0, \psi^1)$ solves

$$\left(\psi^0 \frac{\lambda}{1 + \lambda S_-} + \psi^1 \right) d\langle Y^1, Y^1 \rangle_t = d\langle u, Y^1 \rangle_t$$

and the RHS is determined using Itô's formula.

Conclusion

- Tractable model (to a certain extent)
- Signed variance-minimal martingale measure explicitly determined.
- Extension of Gourieroux, Laurent & Pham (1997) and of Arai (2005).
- Measure transform preserves Markov property, Lévy property is not preserved (Esche & Schweizer 2005).
- PIDE for numerical purposes.
- Restrict to hedging strategies that only depend on S and V.