A characterization of forward utilities

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An agent tries to find the X^* in his set $\mathcal{X}$ of attainable future endowments so that

$$\mathbb{E}[U(X^*)] = \sup_{X \in \mathcal{X}} \mathbb{E}[U(X)]$$

where the utility function $U : \mathbb{R} \mapsto \mathbb{R} \cup \{-\infty\}$ is increasing and concave models the agent's tolerance for risk.

Merton's problem: Suppose the agent can trade in a market $(S_t)_{t\geq 0}$, modelled as a d-dimensional semimartingale.

Fix an initial wealth $X_0 = x$ and let

$$\mathcal{A} = \{\pi = (\pi_t)_{t\geq 0} \text{ predictable} : X_t(\pi) \geq 0 \text{ a.s. for all } t \geq 0\}$$

be the set of admissible strategies, where where

$$X_t(\pi) = x + \int_0^t \pi_s dS_s$$

Fix a time horizon $T > 0$ and let

$$\mathcal{X} = \{X_T(\pi) : \pi \in \mathcal{A}\}$$

Given x, T and the function U, what is

$$\sup_{X \in \mathcal{X}} \mathbb{E}[U(X)]?$$

More importantly, what is the optimizing strategy π^* such that

$$X_T(\pi^*) = X^*?$$

Techniques include

- dynamic programming, Bellman equation
- convex duality, martingales

Example: Suppose $r = 0$ and

$$dS_t = \mathrm{diag}(S_t)(\mu dt + \sigma dW_t)$$

for constants $\mu \in \mathbb{R}^d$ and $\sigma \in \mathbb{R}^{d \times d}$.
Let U have constant relative risk aversion, i.e.,

$$U(x) = \frac{x^\gamma}{\gamma}$$

for some $\gamma < 1, \gamma \neq 0$.

Merton showed

$$\sup_{X \in \mathcal{X}} \mathbb{E}[U(X)] = \frac{x^\gamma}{\gamma} \exp\left(\frac{\gamma}{1-\gamma} \frac{|\sigma^{-1}\mu|^2 T}{2} \right)$$

and the optimal strategy π^* is a scalar multiple of the mutual fund

$$(\sigma\sigma^{\mathrm{T}})^{-1}\mu.$$

In general, however, the optimal strategy π^* may depend on the horizon T.

- For an agent planning for retirement, the horizon T has a clear meaning.
- For a fund manager, who wants to maximize return every day but with no particular terminal date, which T should he use?

Indeed, consider the following two scenarios.

- At time $t = 0$, the investor solves the Merton problem for time horizon $T = 2$ and picks her investment strategy to attain the optimal endowment X^*.
- At time $t = 0$, the investor solves the Merton problem for time horizon $T = 1$ and picks her investment strategy to attain the optimal endowment X'. Then, at time $t = 1$, the investor solves the Merton problem for time horizon $T = 2$, this time with intitial endowment $x = X'$, and picks her investment strategy to attain the optimal endowment X''.

In the two scenarios above, it is generally the case that $X^* \neq X''$ with positive probability.

- In the context of an infinite horizon real option problem, Henderson and Hobson (2006) introduced the notion of horizon-unbiased utility functions.
- Musiela and Zariphopoulou (2006) indroduced the very similar notion of forward dynamic utilities to address the issues raised above.

The idea: Let

$$u(x) = \sup_{X \in \mathcal{X}} \mathbb{E}[U(X)].$$

Then u somehow quantifies the agent's utility at time 0 of initial endowment x.

In more suggestive notation: Let

$$U(x, 0) = \sup_{X \in \mathcal{X}} \mathbb{E}[U(X, T)].$$

Let $\mathcal{X}^x_{t,T}$ set of endowments attainable at time T starting with x at time t.

Definition

A function $U : \mathbb{R} \times [0, \infty) \times \Omega \to \mathbb{R} \cup \{-\infty\}$ is called a forward utility function if

- $x \mapsto U(x, t, \omega)$ is increasing and concave for all $t \geq 0$ and $\omega \in \Omega$
- $$U(x, t) = \sup_{X \in \mathcal{X}^x_{t,T}} \mathbb{E}[U(X, T)|\mathcal{F}_t]$$

  for all $0 \leq t \leq T$.

In the context of Merton's problem, let

$$\mathcal{A}_t^x = \{\pi = (\pi_s)_{s\geq t} \text{ predictable} : X_{s,t}^x(\pi) \geq 0 \text{ a.s. for all } s \geq t\}$$

be the set of admissible strategies, where where

$$X_{s,t}^x(\pi) = x + \int_s^t \pi_u dS_u$$

and

$$\mathcal{X}_{t,T}^x = \{X_{t,T}^x(\pi) : \pi \in \mathcal{A}_t^x\}$$

Question

Do forward dynamic utilities ever exist?

Answer

Yes. For instance, consider

$$dS_t = \mathrm{diag}(S_t)(\mu dt + \sigma dW_t).$$

Then

$$U(x,t) = \frac{x^\gamma}{\gamma} \exp\left(-\frac{\gamma}{1-\gamma} \frac{|\sigma^{-1}\mu|^2 t}{2} \right)$$

Hobson and Henderson, and Musiela and Zariphopoulou have constructed a number of other examples.

Question

If we have a utility function $u : \mathbb{R} \mapsto \mathbb{R} \cup \{-\infty\}$, *can we find a forward utility* U *such that*

$$U(x, 0, \omega) = u(x)?$$

Consider a general market model with

$$dS_t = \mathrm{diag}(S_t)(\mu_t dt + \sigma_t dW_t)$$

where adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$ which may be strictly larger than the filtration generated by W.
Let $\Theta_t = \sigma_t^{-1}\mu_t$ be the market price of risk process. Assume:

- $(\Theta_t)_{t\geq 0}$ is continuous and uniformly bounded.
- $\Theta_t \neq 0$ almost surely for all $t \geq 0$.
- $$\int_0^\infty |\Theta_t|^2 dt = \infty$$

  almost surely.

We only consider forward utilities $U : \mathbb{R} \times [0, \infty) \times \Omega \to \mathbb{R} \cup \{-\infty\}$ that satisfy the following assumptions:

- $U(x, t, \omega) > -\infty$ for all $x > 0$, $t \geq 0$ and $\omega \in \Omega$.
- $x \mapsto U(x, t, \omega)$ is strictly increasing, strictly concave on $(0, \infty)$ for all $t \geq 0$ and $\omega \in \Omega$.
- $(x, t) \mapsto U(x, t, \omega)$ is of class $C^{3,1}$ on $(0, \infty) \times [0, \infty)$ for all $\omega \in \Omega$.
- $\lim_{x \downarrow 0} U_x(x, t, \omega) = \infty$ and $\lim_{x \uparrow \infty} U_x(x, t, \omega) = 0$ for all $t \geq 0$ and all $\omega \in \Omega$, where $U_x = \frac{\partial U}{\partial x}$.
- There exists an optimal endowment $X^* \in \mathcal{X}^x_{t,T}$ such that

$$\mathbb{E}[U(X^*), T) | \mathcal{F}_t] = U(x, t)$$

for each $x > 0$ and $0 \leq t \leq T$.

Answer

A function $u : \mathbb{R}_+ \to \mathbb{R}$ can be extended to a forward dynamic utility such that

$$u(x) = U(x, 0, \omega)$$

if and only if $z \mapsto I(e^z)$ is the Laplace transform of a finite measure, where $I = (u')^{-1}$.

Example: For CARA utility $u(x) = \frac{x^\gamma}{\gamma}$ we have $I(e^z) = \exp\left(z \frac{1}{\gamma - 1}\right)$

Question

How can forward dynamic utilities be characterized?

Answer

Theorem

The following are equivalent:

- *U is a forward utility*
- *There exists a finite measure ν such that*

$$I(y,t) = \int_{(0,\infty)} y^{-r} \exp\left(-\frac{r(r-1)}{2} \int_0^t |\sigma_s^{-1}\mu_s|^2 ds \right) \nu(dr)$$

where $I(\cdot,t) = U_x(\cdot,t)^{-1}$.

Corollary

If U is a forward dynamic utility satisfying the conditions of the theorem, then the optimal portfolio π_t^* is a scalar multiple of the mutual fund

$$(\sigma_t^{\mathrm{T}})^{-1}\Theta_t.$$

Corollary

If U is a forward dynamic utility, then the convex dual function V defined by

$$V(y,t) = \sup_{x>0} U(x,t) - xy$$

satisfies

$$\begin{aligned} V(y,t) &= \inf_{Z \in \mathcal{Z}} \mathbb{E}\left[V\left(y \frac{Z_T}{Z_t}, T \right) \Big| \mathcal{F}_t \right] \\ &= \mathbb{E}\left[V\left(y \frac{Z_T^*}{Z_t^*}, T \right) \Big| \mathcal{F}_t \right] \end{aligned}$$

where

$$Z_t^* = \exp\left(-\frac{1}{2} \int_0^t |\Theta_s|^2 ds - \int_0^t \Theta_s^{-1} dW_s \right)$$

is the density of the Foellmer–Schweizer minimal martingale measure.

Sketch of proof: If U is a forward dynamic utility function, then

$$(U(x,t))_{t\geq 0}$$

is a continuous semimartingale, indexed by x. Hence the Itô-Wentzell formula holds

$$\begin{aligned} U(X_T,T) = U(X_t,t) \quad &+ \quad \int_t^T (\pi_s \cdot \sigma_s^{-1}\mu_s U_x + \frac{|\pi_s \cdot \sigma_s^{-1}|^2}{2} U_{xx}) ds \\ &+ \quad \int_t^T \pi_s \cdot \sigma_s^{-1}\mu_s U_x dW_s \end{aligned}$$

Optimizing yields

$$U_t = \frac{U_x^2}{U_{xx}} \frac{|\Theta|^2}{2}.$$

Hence, the convex dual satisfies

$$V_t = y^2 V_{yy} \frac{|\Theta_t|^2}{2}.$$

But $y \mapsto V(y, t, \omega)$ is decreasing and convex. Together with Widder's characterization of positive space-time harmonic functions yields the direct implication.

Question

Is there an axiomatic foundation for forward utilities a la von Neumann–Morgenstern perference theory?