

Implied volatilities: Where and why not to use them

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The problem

The problem

Basic goal: Construct a dynamic model for

- a stock S
- a bond ($\equiv 1$ in discounted units)
- **several call options** $C(K, T)$ on S

in such a way that the model

- is **arbitrage-free**,
- gives **joint dynamics** of stock and calls,
- is **practically usable**.

Where is the problem ?

Martingale models ?

- Write down model for S , directly under a pricing measure Q .
- Define $C_t(K, T) := E_Q[(S_T - K)^+ | \mathcal{F}_t]$.
- This **martingale model** is obviously arbitrage-free.
- But ...
- ... typically no explicit expressions for $C_t(K, T)$...
- ... **no usable joint dynamics** for S and C ...
- ... and good **calibration can be difficult**.
- This is **no solution !**
- So what now ???

Market models

- **Basic idea:** specify dynamics (SDEs) for **all tradable assets**.
- Joint model for S and all $C(K, T)$ with $K \in \mathcal{K}$ and $T \in \mathcal{T}$.
- **Advantages:**
 - we know **joint dynamics** of S and C by construction.
 - **calibration is automatic** since market prices of options at time 0 are input as initial conditions.
- But: must observe **arbitrage restrictions:**
 - No **static** arbitrage \longrightarrow restrictions on **state space of processes**.
 - No **dynamic** arbitrage \longrightarrow restrictions on **SDE coefficients** (**drift restrictions** à la HJM).
- How to handle these constraints ?

Ideas and questions

The simplest example

- Consider one stock S and **one** call $C(K, T)$. **Restrictions** are
 - **static:** $(S_t - K)^+ \leq C_t \leq S_t$ and $C_T = (S_T - K)^+$.
 - **dynamic:** S and C both martingales under some $Q \approx P$.
- How to write down **SDE for** (S, C) to ensure this ???
- Way out (\rightarrow Lyons 1997, Babbar 2001):
 - **reparametrize:** Instead of C_t , use **implied volatility** $\hat{\sigma}_t$ via $C_t = c_{BS}(S_t, K, (T - t)\hat{\sigma}_t^2)$.
 - more precisely: work with $V_t := (T - t)\hat{\sigma}_t^2$.
 - **static** arbitrage constraint is equivalent to $0 \leq V_t < \infty$ and $V_T = 0$; so **state space** is nice.
 - **dynamic** arbitrage constraint reduces to **drift restriction** for V in SDE model for (S, V) .
 - SDE is still tricky (nonlinear), but feasible; explicit examples.

Multiple maturities

- Consider one stock S and **many** calls $C(K, T)$ with one fixed strike K and **maturities** $T \in \mathcal{T}$.
- Use **new parametrization**:
 - **forward implied volatilities** defined by

$$X_t(T) := \frac{\partial}{\partial T} ((T - t)\hat{\sigma}_t^2(K, T)) = \frac{\partial}{\partial T} V_t(T).$$

- **static** arbitrage constraints are equivalent to
 - $V_t(T) \geq 0$ and $V_T(T) = 0$ as before.
 - $T \mapsto V_t(T)$ is increasing, i.e., $0 \leq X_t(T) < \infty$.
 So: **state space** is nice.
- **dynamic** arbitrage constraints reduce to **drift restrictions** for all $X(T)$ in SDE model for $(S, X(T))_{T \in \mathcal{T}}$.
- \longrightarrow Schönbucher 1999, Brace et al. 2001, Ledoit et al. 2002

Problems with multiple maturities I

- Structure of model: start with

$$\begin{aligned}dS_t &= S_t \mu_t dt + S_t \sigma_t dW_t, \\dX_t(T) &= \alpha_t(T) dt + v_t(T) dW_t.\end{aligned}$$

- **Dynamic arbitrage constraints:** (S_t) and all

$$C_t(T) = c_{BS} \left(S_t, K, \int_t^T X_t(s) ds \right)$$

must be (local) martingales under some $Q \approx P$.

- **Drift restrictions:**

$$\begin{aligned}\mu_t &= -\sigma_t b_t, \\ \sigma_t &= f(X_t(t), S_t, v_t(\cdot)), \\ \alpha_t(T) &= g(X_t(T), S_t, v_t(\cdot), X_t(\cdot)).\end{aligned}$$

Problems with multiple maturities II

- Structure of model with **drift restrictions**:

$$dS_t = S_t f(X_t(t), S_t, v_t(\cdot))(dW_t - b_t dt),$$

$$dX_t(T) = g(X_t(T), S_t, v_t(\cdot), X_t(\cdot)) dt + v_t(T) dW_t.$$

- f and g are **nonlinear**; so even if v is Lipschitz and of linear growth, dt -coefficients and σ are not !
- **Existence problem for (infinite, nonlinear) SDE system !**
- No results in the literature (except classical HJM, with severe conditions: bounded and Lipschitz).

Multiple strikes: even more problems

- Consider one stock S and **many** calls $C(K, T)$ with one fixed maturity T and **strikes** $K \in \mathcal{K}$.
- Arbitrage constraints:
 - **dynamic**: as usual some **drift restrictions**.
 - **static**: $K \mapsto C_t(K, T)$ is convex and satisfies

$$-1 \leq \frac{\partial}{\partial K} C_t(K, T) \leq 0.$$

- **state space** for $C(K, T)$ very complicated.
 - using (classical or forward) implied volatilities does not help either.
- Before even thinking about SDEs: **How to choose parametrization ??**

Results

Infinite SDE systems

- **Key mathematical tool:** consider SDE system

$$dX_t(\theta) = A_t(\theta, X_t(\cdot)) dt + B_t(\theta, X_t(\cdot)) dW_t$$

with $\theta \in \Theta$ (usually $[0, \theta^*]$ or $[0, \infty)$) and $0 \leq t \leq T_0$.

- \longrightarrow **J. Wissel 2006:**
 - **Existence and uniqueness** result for **strong solution** under only **local Lipschitz-type conditions** on A, B .
 - Includes sufficient conditions on growth for non-explosion.
 - Key idea: work on product space $\Theta \times \Omega$.
- **Important:** global Lipschitz condition is too strong for the required applications.

Multiple maturities

- SDE system with **drift restrictions** is

$$dS_t = S_t f(X_t(t), S_t, v_t(\cdot))(dW_t - b_t dt),$$

$$dX_t(T) = g(X_t(T), S_t, v_t(\cdot), X_t(\cdot)) dt + v_t(T) dW_t.$$

- **Theorem:** Sufficient conditions on v (volatilities of forward implied volatilities) for **existence and uniqueness** of solution.
- **Not direct** from general SDE results, because
 - only dW -coefficient v can be chosen here.
 - in addition, must have $X \geq 0$.
- Classes of explicit examples for such models, for **first time in literature**.
- \longrightarrow **S/Wissel 2006**

Explicitly:

$$\begin{aligned} \alpha_t(T) = & -\frac{1}{2} \left((\mathcal{R}_t(T))^2 - \frac{1}{\mathcal{Z}_t(T)} - \frac{1}{4} \right) v_t(T) \cdot \int_t^T v_t(s) ds \\ & + \frac{1}{2} \left((\mathcal{R}_t(T))^2 - \frac{1}{2} \frac{1}{\mathcal{Z}_t(T)} \right) \frac{X_t(T)}{\mathcal{Z}_t(T)} \left| \int_t^T v_t(s) ds \right|^2 \\ & + \left(\mathcal{R}_t(T) - \frac{1}{2} \right) \sigma_t v_t^1(T) \\ & - \mathcal{R}_t(T) \frac{X_t(T)}{\mathcal{Z}_t(T)} \sigma_t \int_t^T v_t^1(s) ds - b_t \cdot v_t(T) \end{aligned}$$

with

$$Y_t(t) := \log S_t, \quad \mathcal{R}_t(T) := \frac{Y_t(t) - \log K}{\mathcal{Z}_t(T)}, \quad \mathcal{Z}_t(T) := \int_t^T X_t(s) ds.$$

Multiple strikes: new parametrization

- Recall key difficulty: **how to parametrize ?**
- Call option prices **admissible** if for each t , $K \mapsto C_t(K)$
 - is C^2 ,
 - is strictly convex,
 - satisfies $-1 < C'_t(K) < 0$ for all K ,
 - satisfies $\lim_{K \rightarrow \infty} C_t(K) = 0$.
- (This is slight strengthening of **static arbitrage constraints**.)
- New concept: local implied volatilities**

$$X_t(K) := \frac{1}{\sqrt{T-t} K C''_t(K)} \varphi\left(\Phi^{-1}\left(-C'_t(K)\right)\right)$$

and, for fixed K_0 , **price level**

$$Y_t := \sqrt{T-t} \Phi^{-1}\left(-C'_t(K_0)\right).$$

- **Theorem:** There is a bijection between **admissible option price models** and all pairs (X, Y) of **positive local implied volatility curves X** and **real-valued price levels Y** .
- In other words:
 - State space of (X, Y) is nice ...
 - ... and yet captures exactly the static arbitrage constraints !
- Also
 - interpretation for the $X_t(T)$ as “local implied volatilities” .
 - explicit formulae relating the classical and the above new local implied volatilities.
 - recovers standard volatility in Black-Scholes setting.
- So: **good solution to parametrization problem with multiple strikes !**

Multiple strikes: structure of models

- **Dynamic arbitrage constraints:**

$$S_t = C_t(0) = \int_0^\infty \Phi \left(\frac{Y_t - \int_{K_0}^k \frac{1}{hX_t(h)} dh}{\sqrt{T-t}} \right) dk$$

and all call prices

$$C_t(K) = \int_K^\infty \Phi \left(\frac{Y_t - \int_{K_0}^k \frac{1}{hX_t(h)} dh}{\sqrt{T-t}} \right) dk$$

must be (local) martingales under some $Q \approx P$.

- **Drift restrictions** on SDEs for X and Y ?

- Model for local implied volatilities X and price level Y :

$$\begin{aligned} dX_t(K) &= X_t(K)u_t(K) dt + X_t(K)v_t(K) dW_t, \\ dY_t &= \beta_t dt + \gamma_t dW_t. \end{aligned}$$

- Drift restrictions** from **dynamic arbitrage constraints**:

$$\begin{aligned} \beta_t &= -\gamma_t \cdot b_t + \frac{1}{2} \frac{Y_t}{T-t} (|\gamma_t|^2 - 1), \\ u_t(K) &= -v_t(K) \cdot b_t + \frac{1}{T-t} \left(\frac{1}{2} (1 - |\gamma_t + \mathcal{I}_t^v(K)|^2) \right. \\ &\quad \left. + (Y_t + \mathcal{I}_t^1(K)) (\gamma_t + \mathcal{I}_t^v(K)) \cdot v_t(K) \right) + |v_t(K)|^2 \end{aligned}$$

with

$$\mathcal{I}_t^1(K) := \int_{K_0}^K \frac{1}{hX_t(h)} dh, \quad \mathcal{I}_t^v(K) := \int_{K_0}^K \frac{v_t(h)}{hX_t(h)} dh.$$

Multiple strikes: existence of models

- **Theorem:** Sufficient conditions on v (volatilities of local implied volatilities) for **existence and uniqueness** of solution.
- Again, **not direct** from general SDE results, because
 - only dW -coefficient v can be chosen here.
 - in addition, must have $X > 0$.
- Up to now, **no result on existence** of such models in the literature.
- First **tractable parametrization** to tackle this problem at all !
- In addition, explicit class of **examples** for models.
- → **S/Wissel 2007**

Towards the end

Open problems (many ...)

- Model and parametrization for **full option surface** (all maturities T and all strikes K): ??
- Practical implementation ?
- Numerical solution ?
- Analogous results for finite family of options ?
→ **some recent progress**
- Recalibration ?
- Markov property ?
- Specific applications ?
- ...

Some related work

- **Dupire: local volatility** model:
 - can also fit any initial term structure of option prices ...
 - ... but seems not rich enough for recalibration over time.
 - no explicit formulas, only PDEs for $C_t(K, T)$ with $t > 0$...
 - ... and hence no joint dynamics for S and C .
 - extensions to more stochastic factors ...
 - ... but there no existence results for models.
- **Bühler:** market models for **variance swaps**:
 - only maturity parameter T ; no strike structure.
 - special payoff function (log) yields easy infinite SDE system.
 - some more explicit results.
- **Durrleman:** links between **spot and implied volatilities**:
 - classical martingale modelling, no market models.
 - results for at-the-money options and shortly before maturity.
 - asymptotic results; but no dynamics for S and C .

References

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The end (for the time being ...)

Thank you for your attention !

<http://www.math.ethz.ch/~mschweiz>