On a simple model for ruin of two insurance companies

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We consider a two-dimensional risk model in which two companies split the amount they pay out of each claim in proportions δ_1 and δ_2 where $\delta_1 + \delta_2 = 1$, and receive premiums at rates c_1 and c_2, respectively. Claims arrive according to a renewal process. We analyse the ruin problems of the risk process leaving the positive quadrant and of entering the negative quadrant under a light-tail assumption.

The multidimensional renewal risk model. In collective risk theory the reserves process X of an insurance company is modeled as

$$X(t) = x + pt - S(t), \tag{1}$$

where x denotes the initial reserve, p is the premium rate per unit of time and $S(t)$ is a stochastic process modeling the amount of cumulative claims up to time t. Taking S to be a compound Poisson or compound renewal process yields the Cramér-Lundberg model and the Sparre-Andersen model, respectively. we consider an extensions of classical risk theory towards a multidimensional reserves model (1) where $X(t), x, p$ and $S(t)$ are vectors, with possible dependence between the components of $S(t)$. Indeed, the assumption of independence of risks may easily fail, for example in the case of reinsurance, when incoming claims have an impact on both insuring companies at the same time.

A general model

Let (U_1, U_2) be a general Lévy process starting at (u_1, u_2).

Questions:

• The exit time from the positive quadrant:

$\tau_{\text{or}}(u_1, u_2) = \inf\{t \geq 0 : U_1(t) < 0 \text{ or } U_2(t) < 0\}$,

• The entrance time into the negative quadrant:

$\tau_{\text{sim}}(u_1, u_2) = \inf\{t \geq 0 : U_1(t) < 0 \,\&\, U_2(t) < 0\}$.

The associated ultimate/perpetual ruin probabilities will be respectively denoted

$$\psi_{\text{or}}(u_1, u_2) = P\left(\tau_{\text{or}}(u_1, u_2) < \infty\right),$$
$$\psi_{\text{sim}}(u_1, u_2) = P\left(\tau_{\text{sim}}(u_1, u_2) < \infty\right).$$

Letting $\tau_i(u_i) = \inf\{t \geq 0 : U_i(t) < 0\}, i = 1, 2$, we also will consider

$\psi_{\text{and}}(u_1, u_2) = P\left(\tau_1(u_1) < \infty \text{ and } \tau_2(u_2) < \infty\right).$

We study asymptotics and exact expressions.

Key-references

Dependent risks/queues
Ambagaspitiya (1999), Müller (1997) Cai and Li (2005) Chan, Yang and Zhang (2003) Denuit, Genest and Marceau (1999) Dhaene and Goovaerts (1996,1997)
Lieshout and Mandjes(2006)

Local limit theorems and large deviations
Arfwedson (1955) Höglund (1990)
Borovkov and Mogulskii (2001) Ignatyuk, Malyshev and Scherbakov (1994)
Collamore (1996)

One-dimensional ruin
Asmussen (2000), Rolski et al. (2003), Knessl and Peters (1994) Pervozvansky (1998)

One-dimensional theory revisited.

Let $\tau(x)$ denote the first passage time of the level $-x$ by Z

$$\tau(x) = \inf\{t \geq 0 : x + Z_t < 0\}$$

where and set

$$\psi(x) = P(\tau(x) < \infty)$$

Let κ be the cumulant: $\kappa(\theta) = \log E[\mathrm{e}^{\theta Z_1}]$.

Let us assume the *Cramér assumption* that there exists a $\gamma > 0$, such that $\kappa(-\gamma) = 0$.

Under this assumption, it holds that

$$\lim_{x \to \infty} \mathrm{e}^{\gamma x} \psi(x) = C, \tag{2}$$

When Z is a Cramér-Lundberg process C is given explicitly by:

$$C = -\kappa'(0)/\kappa'_+(-\gamma). \tag{3}$$

A general model

Let $U = (U_1, U_2)$ be a general Lévy process.

Let the cumulant exponent of U be denoted by κ:

$$\mathrm{e}^{\kappa(\theta_1,\theta_2)t} = E[\mathrm{e}^{\theta_1(U_1(t)-u_1)+\theta_2(U_2(t)-u_2)}]$$

Assume that U has light tails, i.e. there exist $\gamma_i > 0$ s.t.

$$\kappa(-\gamma_1, 0) = \kappa(0, -\gamma_2) = 0.$$

We shall also assume that

$$(-\gamma_1, 0), (0, -\gamma_2) \in \mathrm{int}\{\theta : \kappa(\theta) < \infty\}.$$

Let

$$\psi_i(u_i) = P(\tau_i(u_i) < \infty) = P(\exists t > 0 : U_i(t) < 0)$$

A general model

If $K \to \infty$, it holds that

$$\psi_{\mathrm{or}}(aK, K) \sim \begin{cases} \psi_1(aK) \sim C_1 \mathrm{e}^{-\gamma_1 aK}, & \\ & \text{if } \gamma_1 a < \gamma_2 \\ \psi_1(aK) + \psi_2(K) & \\ & \text{if } \gamma_1 a = \gamma_2 \\ \psi_2(K) \sim C_2 \mathrm{e}^{-\gamma_2 K}, & \\ & \text{if } \gamma_1 a > \gamma_2 \end{cases}$$

and

$$\psi_{\mathrm{and}}(aK, K) = \mathrm{o}(C_2 \mathrm{e}^{-\gamma_2 K} + C_1 \mathrm{e}^{-\gamma_1 aK})$$

Model with co-monotonic dependence

Denote by U_i the risk process of the i'th company

$$U_i(t) := -\delta_i S(t) + c_i t + u_i, \qquad i = 1, 2\ ,$$

where u_i denotes the initial reserve and where

$$S(t) = \sum_{i=1}^{N(t)} \sigma_i, \tag{4}$$

$N(t)$ is a Poisson process with i.i.d. inter-arrival times ζ_i and claims $\sigma_i \geq 0$.

Assume that the second company, receives less premium per amount paid out, i.e.:

$$p_1 = \frac{c_1}{\delta_1} > \frac{c_2}{\delta_2} = p_2.$$

and that

$$p_i > \rho := E\sigma / E\zeta.$$

Geometric considerations

Model with co-monotonic dependence

Write

$$X_i(t) = x_i + p_i t - S(t), \quad i = 1, 2,$$

where

$$S(t) = \sum_{i=1}^{N(t)} \sigma_i$$

with N a Poisson process.

Let κ_i be the cumulant exponent of X_i:

$$E[e^{\theta X_i(t)}] = e^{\kappa_i(\theta) t}, \qquad i = 1, 2. \tag{5}$$

Questions:

- Explicit solutions of $\psi_{\text{and}}, \psi_{\text{or}}, \psi_{\text{sim}}$.

- What are exact asymptotics of $\psi_{\text{and}}, \psi_{\text{or}}, \psi_{\text{sim}}$ if x_1, x_2 go to infinity according to a ray in the positive orthant?

Boundary crossing problems

Write $b_1(t) = x_1 + p_1 t$, $b_2(t) = x_2 + p_2 t$.

Then the problem can also be reformulated as a one-dimensional first-passage problem over a piecewise linear barrier:

$$
\begin{aligned}
\psi_{or}(x_1, x_2) &= P(\exists t > 0 : S(t) > \min\{b_1(t), b_2(t)\}) \\
\psi_{sim}(x_1, x_2) &= P(\exists t > 0 : S(t) > \max\{b_1(t), b_2(t)\}) \\
\psi_{and}(x_1, x_2) &= P(\exists t, s > 0 : S(t) > b_1(t), \\
&\qquad\qquad S(s) > b_2(s))
\end{aligned}
$$

Boundary crossing problems

Let

$$T(x_1, x_2) = \frac{x_2 - x_1}{p_1 - p_2}.$$

be the crossing time of b_1 and b_2 and write $\tau_i(x) = \inf\{t \geq 0 : X_i(t) < 0\}$.

Proposition 1 *Suppose that* $x_2 > x_1, p_2 < p_1$*. The ruin probabilities* $\psi_{\text{and}}, \psi_{\text{sim}}$ *and* ψ_{or} *are given by:*

$$\begin{aligned}
\psi_{\text{sim}}(x_1, x_2) &= P(\tau_2(x_2) \leq T) \\
&+ P(\tau_2(x_2) > T, \inf_{s>T} X_1(s) < 0), \\
\psi_{\text{or}}(x_1, x_2) &= P(\tau_1(x_1) \leq T) \\
&+ P(\tau_1(x_1) > T, \inf_{s>T} X_2(s) < 0), \\
\psi_{\text{and}}(x_1, x_2) &= P(T < \tau_1(x_1) < \infty) \\
&+ P(\tau_1(x_1) \leq T, \tau_2(x_2) < \infty).
\end{aligned}$$

Exact solution

Let $q^+(\alpha)$ the largest root of

$$\kappa_1(q) = \alpha$$

and write

$$\overline{\psi}_{or}(x_1, x_2) = 1 - \psi_{or}(x_1, x_2)$$

Corollary 2 *The double Laplace transform*

$$\tilde{\psi}_{or}(p, q) = \int_0^\infty \int_0^\infty e^{-px_1} e^{-qx_2}\, \overline{\psi}_{or}(x_1, x_2)\, dx_1\, dx_2$$

is given by

$$\tilde{\psi}_{or}(p, q) = \frac{\kappa_1'(0+) + (p_2 - p_1)}{p(\kappa_1(p+q) - q(p_1 - p_2))} \times \left[1 + \frac{p}{q - q^+(q(p_1 - p_2))}\right]$$

Example. If S is compound Poisson with Exp(μ) jumps then $\tilde{\psi}$ is given by

$$\tilde{\psi}(p,q) = \frac{(\mu + p + q)(p_2 - \rho)}{pp_1(z_1(q) - p)z_2(q)}$$

where $\rho = \lambda/\mu$ and $z_1(q) < z_2(q)$ are the roots of $\kappa_1(p+q) = (p_1 - p_2)q$.

Example. Small claims model If

$$X_i(t) = x_i + p_i t - B(t)$$

where B is a Brownian motion and $x_2 > x_1$, then:

$$\begin{aligned}
&\psi_{\text{or}}(x_1, x_2) \\
&= 1 - \Phi\left(a(x_1, p_1)\right) + \mathrm{e}^{-2p_1x_1}\Phi\left(a(-x_1, p_1)\right) \\
&+ \mathrm{e}^{-2p_2x_2} \times [\Phi\left(a(x_1, p_1 - 2p_2)\right) \\
&- \mathrm{e}^{-2x_1(p_1-2p_2)}\Phi\left(a(-x_1, p_1 - 2p_2)\right)].
\end{aligned}$$

$$\begin{aligned}
&\psi_{\text{sim}}(x_1, x_2) \\
&= 1 - \Phi\left(a(x_2, p_2)\right) + \mathrm{e}^{-2p_2x_2}\Phi\left(a(-x_2, p_2)\right) \\
&+\mathrm{e}^{-2p_1x_1} \times [\Phi\left(a(x_2, p_2 - 2p_1)\right) \\
&-\mathrm{e}^{-2x_2(p_2-2p_1)}\Phi\left(a(-x_2, p_2 - 2p_1)\right)]
\end{aligned}$$

where

$$a(x, p) = [x + pT]/\sqrt{T}$$

and Φ denotes the cumulative standard normal distribution function.

Asymptotic cones

$$\begin{aligned}\mathcal{D}_1 &= \{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 > x_2 s_1\},\\ \mathcal{D}_2 &= \{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 < x_2 s_2\}\end{aligned}$$

with

$$s_1 = \frac{\kappa_1'(-\gamma_1)}{\kappa_2'(-\gamma_1)}, \qquad s_2 = \left(\frac{\kappa_1'(-\gamma_2)}{\kappa_2'(-\gamma_2)}\right)_+$$

and

$$\mathcal{D}_0 = \mathbb{R}^2_+ \backslash [\overline{\mathcal{D}}_1 \cup \overline{\mathcal{D}}_2]$$

where $\overline{\mathcal{D}}_i$ is the closure of $\mathcal{D}_i$, the open cone lying between $\mathcal{D}_1$ and $\mathcal{D}_2$. Write

$$\widehat{D}_2 = \{(x_1, x_2) : x_1 < x_2 s_3\}$$

where $s_3 = \kappa_1'(-\widehat{\gamma}_2)/\kappa_2'(-\widehat{\gamma}_2)$ with $\widehat{\gamma}_2$ is the largest root of

$$\kappa_1(-s) = \kappa_1(-\gamma_2).$$

Lemma 3 *1.* $\mathcal{D}_i, i = 0, 1, 2$ *are disjoint and* $\mathcal{D}_0, \mathcal{D}_1 \neq \emptyset$.

2. $\mathcal{D}_1 \subset \mathcal{U} := \{(x_1, x_2) \in \mathbb{R}_+^2 : x_2\gamma_2 < x_1\gamma_1\}$

3. $\mathcal{D}_2 \neq \emptyset$ *iff* $\kappa_1'(-\gamma_2) < 0$.

Lemma 4 *Writing* $i = 1, 2$

$$T_i = x_i/[-\kappa_i'(-\gamma_i)]$$

and

$$\widetilde{T}_i = x_i/[-\kappa_i'(-\gamma_{3-i})]$$

and $a(x) = \frac{x_1}{x_2}$ *(where* $x = (x_1, x_2) \in \mathbb{R}_+^2$*), the following hold true:*

$$\begin{aligned}
\mathcal{D}_2 &= \{x \in \mathbb{R}_+^2 : a(x) < s_2\} = \{x \in \mathbb{R}_+^2 : \widetilde{T}_1 < T\} \\
&= \{x \in \mathbb{R}_+^2 : T_2 < T\} \\
\mathcal{D}_0 &= \{x \in \mathbb{R}_+^2 : T_1 < T < T_2\} \\
\mathcal{D}_1 &= \{x \in \mathbb{R}_+^2 : s_1 < a(x)\} = \{x \in \mathbb{R}_+^2 : T < \widetilde{T}_2\} \\
&= \{x \in \mathbb{R}_+^2 : T < T_1\}.
\end{aligned}$$

Asymptotics

Notation. • κ_2 denotes the characteristic exponent of X_2 and

$$\kappa^*(v) = \kappa_2^*(v) = \sup_{\beta \in \mathbb{R}} [v\beta - \kappa_2(\beta)]. \qquad (6)$$

the convex conjugate of κ_2

• Write

$$\underline{\theta} = \inf\{\theta : \kappa_2(\theta) < \infty\} \qquad \underline{v} = \lim_{\theta \downarrow \underline{\theta}} \kappa_2'(\theta)$$

• **The adjustment coefficient.**

Assume there exist $\gamma_i > 0$, such that

$$\kappa_i(-\gamma_i) = 0$$

and let

$$\gamma(a) = \kappa_2^*(-v_a)/v_a,$$

where

$$v_a := (p_1 - p_2)/(1 - a).$$

Asymptotics

Set

$$\overline{a} = 1 + \frac{(p_1 - p_2)}{\underline{v}}$$

Theorem. Assume that

$$\underline{\theta} < -\gamma_1.$$

For $a < \overline{a}$, it holds as $K \to \infty$,

$$\psi_{\mathsf{sim}}(aK, K) \quad \sim \quad \begin{cases} C_1 \mathrm{e}^{-\gamma_1 a K} \\ \qquad\qquad \text{if } (aK, K) \in \mathcal{D}_1 \\ (D^{\#}(v_a) + D'(v_a))K^{-1/2}\mathrm{e}^{-\gamma(a)K} \\ \qquad\qquad \text{if } (aK, K) \in \mathcal{D}_0 \\ C_2 \mathrm{e}^{-\gamma_2 K} \\ \qquad\qquad \text{if } (aK, K) \in \mathcal{D}_2, \end{cases}$$

for some $D^{\#}, D' > 0$ and $C_i = -\kappa_i'(0)/\kappa_i'(-\gamma_i)$.

Constants

Denote by $\theta_v < \theta_v'$ the roots

$$\kappa_2'(\theta_v) = -v$$

and

$$\kappa_2(\theta_v) = \kappa_2(\theta_v')$$

then

$$\begin{aligned} D'(w) &= \frac{\theta_w' - \theta_w}{|\theta_w \theta_w'|} \frac{\sqrt{w}}{\sqrt{2\pi\kappa_1''(\theta_w)}}, \\ D^{\#}(w) &= \left[\frac{1}{\theta_w} - \frac{1}{\theta_w'} + \frac{\kappa_1'(0)}{\kappa_1(\theta_w')} - \frac{\kappa_1'(0)}{\kappa_1(\theta_w)}\right] \frac{\sqrt{w}}{\sqrt{2\pi\kappa_1''(\theta_w)}} \end{aligned}$$

Asymptotics
Write

$$\widehat{D}_2 = \{(x_1, x_2) : x_1 < x_2 s_3\}$$

where $s_3 = \kappa_1'(-\widehat{\gamma}_2)/\kappa_2'(-\widehat{\gamma}_2)$ with $\widehat{\gamma}_2$ is the largest root of

$$\kappa_1(-s) = \kappa_1(-\gamma_2).$$

Theorem. For $v_a < -\underline{v}$ it holds that, as $K \to \infty$,

$$\psi_{\text{and}}(aK, K) \sim \begin{cases} C_1 \mathrm{e}^{-\gamma_1 aK} \\ \qquad\qquad \text{if } (aK, K) \in \mathcal{D}_1 \\ \\ (D^\star(v_a) - D^\dagger(v_a))K^{-1/2}\mathrm{e}^{-\gamma(a)K} \\ \qquad\qquad \text{if } (aK, K) \in \widehat{\mathcal{D}}_0 \\ \\ \widehat{C}_2 \mathrm{e}^{-(a\widehat{\gamma}_2 + (1-a)\gamma_2)K} \\ \qquad\qquad \text{if } (aK, K) \in \widehat{\mathcal{D}}_2, \end{cases}$$

for some $D^\star, D^\dagger > 0$ where

$$\widehat{C}_2 = C_2 \kappa_1'(-\gamma_2)/\kappa_1'(-\widehat{\gamma}_2).$$

Constants.

Denote by $\theta_v^\star > \theta_v$ the root of

$$\kappa_1(\theta_v) = \kappa_1(\theta_v^\star)$$

then $D^\star(w)$ and $D^\dagger(w)$ are respectively given by with θ_w' replaced by $\theta_w^\star$ and κ_1 replaced by κ_2.

One-dimensional theory revisited.

Write $f(x) \sim h(x)$ $(x \to \infty)$ if $\lim f/h(x) = 1$ and

$$w(x,t) = \psi(x) - \psi(x,t) = P(t < \tau(x) < \infty).$$

Höglund-Arfwedson Theorem

Assume that either $\kappa'_+(0) < 0$ or that the Cramér assumption holds and write $\zeta = \min\{\theta : \kappa(\theta) = 0\}$. If $0 < v < -\underline{v}$ and $x, t \to \infty$ such that $x/t = v$, it holds that

$$\psi(x,t) \sim \begin{cases} C\mathrm{e}^{-x\zeta} & \text{if } x/t < -\kappa'_+(-\zeta) \\ |D(v)|t^{-1/2}\mathrm{e}^{-t\kappa^*(-v)} & \text{if } x/t > -\kappa'_+(-\zeta), \end{cases}$$

$$w(x,t) \sim \begin{cases} |D(v)|t^{-1/2}\mathrm{e}^{-t\kappa^*(-v)} & \text{if } x/t < -\kappa'_+(-\zeta) \\ C\mathrm{e}^{-x\zeta} & \text{if } x/t > -\kappa'_+(-\zeta), \end{cases}$$

with

$$D(v) = c(v) \cdot \frac{1}{\sqrt{2\pi\kappa''(\theta_v)}}, \qquad c(v) = \frac{\theta'_v - \theta_v}{\theta_v \theta'_v},$$

Asymptotic Limit laws

The following result shows existence of and identifies the limit laws of the Cramér-Lundberg process $X(t)$ conditioned on $t < \tau$ and on $t > \tau$:

Theorem Assume that $\underline{\theta} < 0$ and $\kappa'(0) < 0$.

(i) If $0 < v < -\kappa'(0)$,

$$\overline{\Psi}_v = \lim_{x,t\to\infty, x=tv} P(X(t) \in \cdot\ |\tau(x) > t),,$$

in the sense of weak convergence, where

$$\overline{\Psi}_v(\mathrm{d}y) = c(v)^{-1}[\mathrm{e}^{-\theta_v y} - \mathrm{e}^{-\theta'_v y}]\mathbf{1}_{(0,\infty)}(y)\mathrm{d}y.,$$

(ii) If $-\kappa'(0) < v < -\underline{v}$,

$$\Psi_v = \lim_{x,t\to\infty, x=tv} P(X(t) \in \cdot\ |\tau(x) < t)$$

in the sense of weak convergence, where

$$\Psi_v(\mathrm{d}y) = |c(v)|^{-1}[\mathrm{e}^{-\theta'_v y}\mathbf{1}_{(0,\infty)}(y) + \mathrm{e}^{-\theta_v y}\mathbf{1}_{(-\infty,0)}(y)]\mathrm{d}$$

Two term expansions

We write $f \approx g + h$ if $\lim_{x\to\infty}(f-g)/h(x) = \lim_{x\to\infty}(f-h)/g(x) = 1$.

Proposition If $x_1, x_2 \to \infty$ such that $x_2/T(x_1,x_2) = v$, $v \neq -\kappa_2'(-\gamma_i)$ $i = 1,2$, it holds that

$$\begin{aligned} \psi_{\text{sim}}(x_1,x_2) &\approx& \psi_2(x_2,T) \\ &+& \widetilde{C}_1(v)\mathrm{e}^{-\gamma_1 x_1}\overline{\psi}_2^{(-\gamma_1)}(x_2,T), \end{aligned}$$

for some $C_1(v)$. If S is a compound Poisson process, it holds for $i = 1,2$,

$$\widetilde{C}_i(v) = \begin{cases} C_i & \\ & \text{if } -\kappa_2'(-\gamma_i) < v \\ c_{3-i}(v,\gamma_i)^{-1}[\psi_i^*(\theta_v) - \psi_i^*(\eta_v^{(i)})] & \\ & \text{if } 0 < v < -\kappa_2'(-\gamma_i), \end{cases}$$

and $\eta_v^{(1)} = \theta_v'$, $\eta_v^{(2)} = \theta_v^\star$.

Renewal model – Asymptotics

$$S(t) = \sum_{i=1}^{N(t)} \sigma_i, \tag{7}$$

where $N(t)$ is a renewal process with i.i.d. inter-arrival times ζ_i, and the claims σ_i are i.i.d. positive random variables independent of $N(t)$.

Cramer/light tail assumptions

Assume that there exist constants $\gamma_i > 0$ ($i = 1, 2$) such that

$$E[\mathrm{e}^{\gamma_i(\sigma_1 - p_i\zeta_1)}] = 1. \tag{8}$$

Moreover, assume that, for some $\epsilon > 0$,

$$E[\mathrm{e}^{(\gamma_1+\epsilon)\sigma_1}] < \infty. \tag{9}$$

Theorem 5 *Let $a < 1$. Under the Cramer conditions it holds that, as $K \to \infty$,*

$$\begin{aligned} \psi_{\mathrm{or}}(aK, K) &\sim C_2\mathrm{e}^{-\gamma_2 K} + C_1\mathrm{e}^{-\gamma_1 aK}, \\ \psi_{\mathrm{and}}(aK, K) &= \mathrm{o}(C_2\mathrm{e}^{-\gamma_2 K} + C_1\mathrm{e}^{-\gamma_1 aK}). \end{aligned}$$