Credit Risk Models Based on Time Changed Brownian Motion

Tom Hurd

Mathematics and Statistics
McMaster University

Further Developments in Quantitative Finance, ICMS Edinburgh

Toronto: the place to be in 2010 !

- Where are you going to be in 2010 ?

Toronto: the place to be in 2010 !

- Where are you going to be in 2010 ?
- The Fields Institute is going to host a thematic program in *Mathematical Finance*

Toronto: the place to be in 2010 !

- Where are you going to be in 2010 ?
- The Fields Institute is going to host a thematic program in *Mathematical Finance*
- Activities from January to July 2010 include: 3 major workshops, 4-6 weekend meetings, 3 graduate courses.

Toronto: the place to be in 2010 !

- Where are you going to be in 2010 ?
- The Fields Institute is going to host a thematic program in *Mathematical Finance*
- Activities from January to July 2010 include: 3 major workshops, 4-6 weekend meetings, 3 graduate courses.
- Main themes are: *mathematical foundations*, *computational finance*, and *emerging applications* .

Toronto: the place to be in 2010 !

- Where are you going to be in 2010 ?
- The Fields Institute is going to host a thematic program in *Mathematical Finance*
- Activities from January to July 2010 include: 3 major workshops, 4-6 weekend meetings, 3 graduate courses.
- Main themes are: *mathematical foundations*, *computational finance*, and *emerging applications* .
- Opportunities for long and short term visitors, postdocs, graduate students, etc.

Toronto: the place to be in 2010 !

- Where are you going to be in 2010 ?
- The Fields Institute is going to host a thematic program in *Mathematical Finance*
- Activities from January to July 2010 include: 3 major workshops, 4-6 weekend meetings, 3 graduate courses.
- Main themes are: *mathematical foundations*, *computational finance*, and *emerging applications* .
- Opportunities for long and short term visitors, postdocs, graduate students, etc.
- Contact the organizers: TRH and Matheus Grasselli (McMaster), M. Rindesbacher (U of T), V. Henderson (Warwick), Y. Ait-Sahalia (Princeton).

Two outstanding issues in credit risk

1. For one firm, how can one flexibly model default dynamics (bonds and other debt securities) jointly with equity dynamics (stocks and equity options), consistent with arbitrage pricing theory?

Two outstanding issues in credit risk

1. For one firm, how can one flexibly model default dynamics (bonds and other debt securities) jointly with equity dynamics (stocks and equity options), consistent with arbitrage pricing theory?
2. How can one put these one firm models together efficiently to price large scale portfolio credit products like "Collateralized Debt Obligations"?

Black-Cox (1976)

A time-consistent modification of Merton's 1974 model:

1. Value of a firm: $V_t = \exp[\log V_0 + \sigma W_t + (\mu - \frac{1}{2}\sigma^2)t]$
2. Moving debt barrier: $K(t) = \exp[\log K_0 + \gamma t]$;
3. Log-leverage ratio:
   $L_t = \frac{1}{\sigma}\log(V_t/K(t)) = L_0 + W_t + (\mu - \frac{1}{2}\sigma^2 - \gamma)t$;
4. Default occurs at first time $L_t \le 0$;
5. First passage time:

$$t^* = t^*_{a,b} := \inf\{t | W_t + a - bt \le 0\}$$

   where $a = L_0$ and $b = \frac{1}{\sigma}(\frac{1}{2}\sigma^2 + \gamma - \mu)$.

6. This leads to the famous formula:

$$P[t^*_{a,b} \le t] = F(a,b,t) := \Phi\left(\frac{-a+bt}{\sqrt{t}}\right) + e^{-2ba}\Phi\left(\frac{-a-bt}{\sqrt{t}}\right)$$

Difficulties with Black-Cox

1. Defaults are predictable, which implies incorrect short time behaviour;
2. The extension to M firms is very difficult: the distribution of first passage times for correlated Brownian Motions is basically an unsolvable problem! Zhou (2001) solves it for $M = 2$, but higher M seems unreachable.
3. Rigid structure is hard to fit to real data.

Is there another way?

1. Reduced form models beginning with Jarrow-Turnbull (1995) have unpredictable defaults with an exogeneously given instantaneous hazard rate;
2. Jarrow-Lando-Turnbull 1997 model the credit rating as a Markov chain with stochastic transition probabilities. This model interpolates between structural and reduced form frameworks.
3. Lando (1998), Arvanitis-Gregory-Laurent (1999), Hurd-Kuznetsov (2007) give computationally efficient versions of JLT.
4. Carr-Wu (2005) give reduced form models with the hazard rate tied to the level of the stock price.

Jump-Diffusion Stock Models

Many famous stock models can be written in terms of time-changed drifting Brownian Motion.

1. Log stock process $\log S_t := s_t = s_0 + [W_{G_t} + \beta G_t]$ for some increasing process G_t independent of W.
2. For example, the Variance Gamma model of Madan et al takes G_t as a Gamma process with two parameters.
3. Other Lévy models: NIG, Meixner, hyperbolic, etc.
4. Stochastic volatility models: $G_t = \int_0^t \lambda_s ds, \lambda_t \sim \mathrm{CIR}$.
5. In cases of interest, the characteristic function $\Phi_G(u,t) = E[e^{iuG_t}]$ is explicit.
6. Further flexibility: if $G = G^{(1)} + G^{(2)}$ where $G^{(1)}, G^{(2)}$ are independent, then

$$\Phi_G(u,t) = \Phi_{G^{(1)}}(u,t)\Phi_{G^{(2)}}(u,t)$$

Jump-Diffusion Default Models

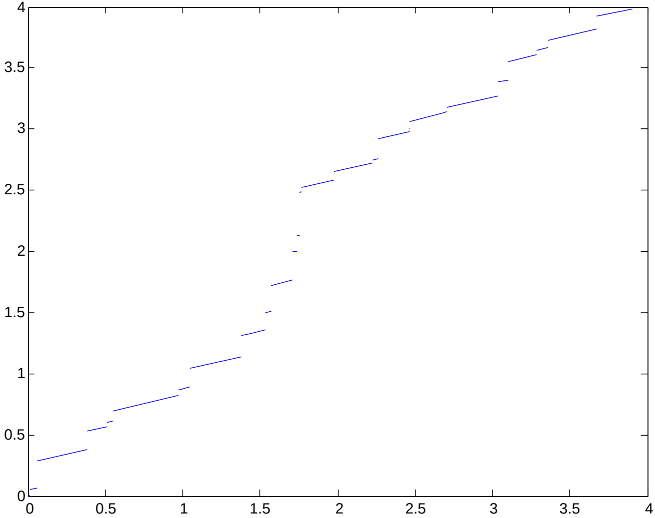
1. Rather than the stock price, instead focus on the log-leverage ratio

$$L_t = \mathsf{log}\left(\frac{V_t}{K(t)}\right)$$

2. Write it as the time-change of arithmetic drifting Brownian Motion:

$$L_t = X_{G_t}, \quad X_t = x + W_t + \beta t$$

A sample path of Lévy time change G_t



First passage times for time-changed BM

1. The usual extension of the Black-Cox model would be to define default time to be the first passage time:

$$t^{(1)} = \mathsf{inf}\{t | L_t \leq 0\}$$

2. This has been well studied, and is difficult to compute!
3. Here is an alternative:

$$t^{(2)} = \mathsf{inf}\{t | G_t \geq t^*\}$$

   where

$$t^* = \mathsf{inf}\{t | X_t \leq 0\}$$

4. This is relatively easy to compute.
5. The natural stopped process is $L_t^{(2)} := X_{G_t \wedge t^*}$.

Bond Pricing Formulas

1. Default probability before time t:

$$\begin{aligned} P(t,x) &= P[t^{(2)} \le t] = E[P[t^* \le y | G_t = y]] \\ &= \int_0^\infty F(x,\beta,y)\rho_t(y)dy \\ &= \frac{1}{2\pi}\int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{e^{-x[\beta+\sqrt{\beta^2+2iu}]}}{iu}\Phi_G(u,t)du \end{aligned}$$

   where ρ_t is the PDF of G_t. $\epsilon \in (0,\bar{\epsilon})$ is a regularization parameter.

2. Price of zero coupon bond with maturity T and fractional recovery R:

$$\bar{B}^{RT}(T) = e^{-rT}[R + (1-R)P(T,x)]$$

3. One can also price bond options in close-to-closed form.

Example

1. Take $L_0 = 1$;
2. Take $b = 0.09$, so the stock volatility is approximately 30%;
3. Take G to have Lévy distribution $\nu(y) = cae^{-ay}$ with jump intensity $c = 0.01$ and mean jump size a^{-1}.

Then the hazard rate for default (the instantaneous default rate) is

$$h_0 = c\left(\frac{V_0}{D(0)}\right)^{-a}$$

Numerics

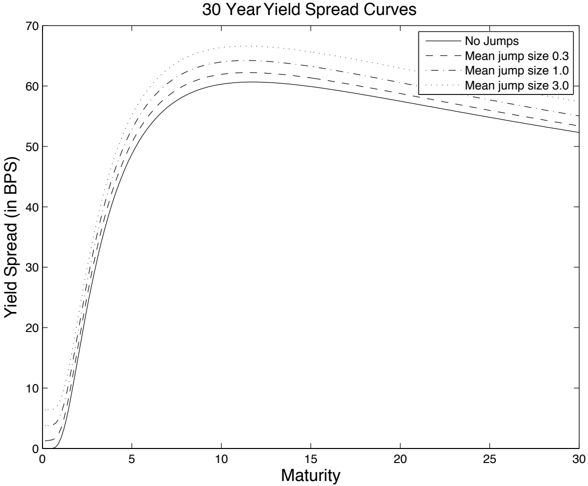


Figure: Thirty year yield spreads in basis points, $c = 0.01$ and $a^{-1} = 0.30, 1.0, 3.0$.

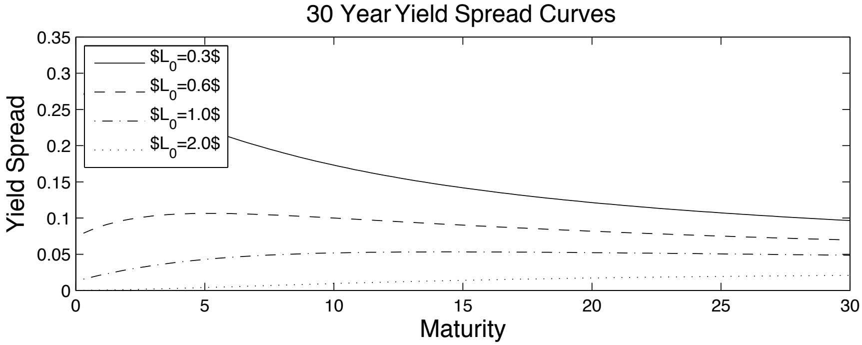


Figure: Thirty year yield spreads with four different values $L_0 = 0.3, 0.6, 1.0, 2.0$.

A Simple Multifirm Model

1. Take independent drifting BMs $\{X^1, X^2, \dots, X^M\}$, and parameters $\{x_i, \beta_i\}$
2. Subject them to the same time-change G_t:

$$L^i(t) = X^i_{G_t}$$

3. Moments:

$$\begin{aligned}
\text{Mean} &= E[L^i_t] = x_i + \beta_i t E[G_1]/2 \\
\text{Variance} &= \mathrm{Var}[L^i_t] = t(E[G_1] + (\beta^i)^2 \mathrm{Var}[G_1]) \\
\text{Correlation } \rho_{ij} &= \left(1 + \frac{E[G(1)]}{\beta_i^2 \mathrm{Var}[G(1)]}\right)^{-1/2} \left(1 + \frac{E[G(1)]}{\beta_j^2 \mathrm{Var}[G(1)]}\right)^{-1/2}
\end{aligned}$$

Computing the Joint Default Distribution

1. $P(\Sigma, t)$, the probability that the default subset is $\Sigma \subset \{1, \dots, M\}$, is:

$$P(\Sigma, t) = \int_0^\infty \left[\prod_{i\in\Sigma} F(x^i, \beta^i, g)\right]\left[\prod_{i\notin\Sigma}(1 - F(x^i, \beta^i, g))\right]\rho_t(g)dg$$

2. Finally, $P[N_t \geq k]$ is given by

$$\sum_{\Sigma:|\Sigma|\geq k} P(\Sigma, t)$$

3. Main Conclusion: Subjecting M independent Brownian Motions to an identical time-change process leads to one-dimensional integral!

Joint Equity-Default Models

Here is a simple "one–factor" model that extends stock price models like the VG and NIG models to include default:

1. Debt per share at time t: $D(t) = e^{rt} D_0$;
2. Pre-default share price (equity per share) at time t

$$S_t + D(t) = D(t)e^{L_t}$$

3. Jump-diffusion dynamics with $X_t = x + W_t + \beta t$ and time change G:

$$L_t = \log(V_t / D(t)) = X_{G_t}$$

4. Default is the first time G_t exceeds t^*, which is also the first time $S(t) = 0$.
5. Healthy firm: $S_t \gg D(t)$, so $S_t \sim D(t)e^{L_t}$ is approximately the usual "geometric" dynamics.
6. Unhealthy firm: $S_t \lesssim D(t)$, dynamics is "arithmetic".

Risk Neutral Dynamics

Suppose W_t is a BM. Note

1. e^{X_t} is a $\sigma(W)$-martingale if $X_t = x + W_t - t/2$;

Risk Neutral Dynamics

Suppose W_t is a BM. Note

1. e^{X_t} is a $\sigma(W)$-martingale if $X_t = x + W_t - t/2$;
2. $e^{X_{t \wedge t^*}}$ is a $\sigma(W)$-martingale for any $\sigma(W)$-stopping time t^* (by Optional Sampling);

Risk Neutral Dynamics

Suppose W_t is a BM. Note

1. e^{X_t} is a $\sigma(W)$-martingale if $X_t = x + W_t - t/2$;
2. $e^{X_{t\wedge t^*}}$ is a $\sigma(W)$-martingale for any $\sigma(W)$-stopping time t^* (by Optional Sampling);
3. $e^{L_t^{(2)}}$ is a $\mathcal{F}_t$-martingale, where $L_t^{(2)} := X_{G_t \wedge t^*}$ is the time-changed stopped martingale.

Risk Neutral Dynamics

Suppose W_t is a BM. Note

1. e^{X_t} is a $\sigma(W)$-martingale if $X_t = x + W_t - t/2$;
2. $e^{X_{t\wedge t^*}}$ is a $\sigma(W)$-martingale for any $\sigma(W)$-stopping time t^* (by Optional Sampling);
3. $e^{L_t^{(2)}}$ is a $\mathcal{F}_t$-martingale, where $L_t^{(2)} := X_{G_t \wedge t^*}$ is the time-changed stopped martingale.
4. Therefore $e^{-rt}S_t = D_0(e^{L_t^{(2)}} - 1)$ is a $\mathcal{F}_t$-martingale;

Risk Neutral Dynamics

Suppose W_t is a BM. Note

1. e^{X_t} is a $\sigma(W)$-martingale if $X_t = x + W_t - t/2$;
2. $e^{X_{t\wedge t^*}}$ is a $\sigma(W)$-martingale for any $\sigma(W)$-stopping time t^* (by Optional Sampling);
3. $e^{L_t^{(2)}}$ is a $\mathcal{F}_t$-martingale, where $L_t^{(2)} := X_{G_t \wedge t^*}$ is the time-changed stopped martingale.
4. Therefore $e^{-rt}S_t = D_0(e^{L_t^{(2)}} - 1)$ is a $\mathcal{F}_t$-martingale;
5. The model is arbitrage-free!

Risk Neutral Dynamics

Suppose W_t is a BM. Note

1. e^{X_t} is a $\sigma(W)$-martingale if $X_t = x + W_t - t/2$;
2. $e^{X_{t\wedge t^*}}$ is a $\sigma(W)$-martingale for any $\sigma(W)$-stopping time t^* (by Optional Sampling);
3. $e^{L_t^{(2)}}$ is a $\mathcal{F}_t$-martingale, where $L_t^{(2)} := X_{G_t \wedge t^*}$ is the time-changed stopped martingale.
4. Therefore $e^{-rt} S_t = D_0(e^{L_t^{(2)}} - 1)$ is a $\mathcal{F}_t$-martingale;
5. The model is arbitrage-free!
6. S_t first hits zero at precisely $t^{(2)}$, and remains zero thereafter.

Equity-Credit Computations

Pricing credit derivatives is identical to before. Equity derivatives are also (almost) explicit:

Proposition

A call option on the stock with maturity T and strike K

$$C(S_0, T, K) = E^Q[e^{-rT}(S_T - K)^+ \mathbf{1}_{\{t^{(2)} > T\}}]$$

has price $C(S_0, T, K) = D_0 f(T, \log(1 - S_0/D_0), \log(1 + e^{-rT}K/D_0))$, *where one has a convergent Fourier integral*

$$\begin{aligned} f(T, x, k) &= E_x[E[(e^{L_T} - e^k)^+ | G_T]] \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{iuk - k(\alpha - 1)}}{(\alpha - 1 - iu)(\alpha - iu)} e^{iu(x + \beta t) - u^2 t/2} \\ &\times \left[N\left(\frac{x + (\beta + iu)t}{\sqrt{t}}\right) - e^{-2(\beta + iu)x} N\left(\frac{-x + (\beta + iu)t}{\sqrt{t}}\right)\right] du \end{aligned}$$

for any $\alpha \in (1, \alpha_{\max})$.

Conclusion

1. By taking $L_t = X_{G_t}$ with different time changes G_t, one has a rich variety of generalized Black-Cox models;

Conclusion

1. By taking $L_t = X_{G_t}$ with different time changes G_t, one has a rich variety of generalized Black-Cox models;
2. With the simple idea of shared time-change, an interesting and computable "one-factor" multifirm correlation structure emerges.

Conclusion

1. By taking $L_t = X_{G_t}$ with different time changes G_t, one has a rich variety of generalized Black-Cox models;
2. With the simple idea of shared time-change, an interesting and computable "one-factor" multifirm correlation structure emerges.
3. The extension to include equity is compatible with well-studied stock price models like the VG, NIG and stochastic volatility models;

Conclusion

1. By taking $L_t = X_{G_t}$ with different time changes G_t, one has a rich variety of generalized Black-Cox models;
2. With the simple idea of shared time-change, an interesting and computable "one-factor" multifirm correlation structure emerges.
3. The extension to include equity is compatible with well-studied stock price models like the VG, NIG and stochastic volatility models;
4. With $t^{(2)}$, the martingale condition for S_t is explicitly verifiable;

Conclusion

1. By taking $L_t = X_{G_t}$ with different time changes G_t, one has a rich variety of generalized Black-Cox models;
2. With the simple idea of shared time-change, an interesting and computable "one-factor" multifirm correlation structure emerges.
3. The extension to include equity is compatible with well-studied stock price models like the VG, NIG and stochastic volatility models;
4. With $t^{(2)}$, the martingale condition for S_t is explicitly verifiable;
5. Our joint equity-credit model takes into account the influence of stock value on default hazard rates.

Conclusion

1. By taking $L_t = X_{G_t}$ with different time changes G_t, one has a rich variety of generalized Black-Cox models;
2. With the simple idea of shared time-change, an interesting and computable "one-factor" multifirm correlation structure emerges.
3. The extension to include equity is compatible with well-studied stock price models like the VG, NIG and stochastic volatility models;
4. With $t^{(2)}$, the martingale condition for S_t is explicitly verifiable;
5. Our joint equity-credit model takes into account the influence of stock value on default hazard rates.
6. Computations are based intensively on Fast Fourier Transform;

Conclusion

1. By taking $L_t = X_{G_t}$ with different time changes G_t, one has a rich variety of generalized Black-Cox models;
2. With the simple idea of shared time-change, an interesting and computable "one-factor" multifirm correlation structure emerges.
3. The extension to include equity is compatible with well-studied stock price models like the VG, NIG and stochastic volatility models;
4. With $t^{(2)}$, the martingale condition for S_t is explicitly verifiable;
5. Our joint equity-credit model takes into account the influence of stock value on default hazard rates.
6. Computations are based intensively on Fast Fourier Transform;
7. Once the FFT is safely working and a good option/CDS data set is obtained, risk-neutral calibration should be straightforward.