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Pricing and hedging of derivatives based on non-tradable underlyings

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Paper on:

http://www.mathematik.hu-berlin.de/ $\sim$ ankirchn/

Examples for non-tradable underlyings

- basket indices
- weather indices: temperature, rainfall of snowfall
- loss indices
- prices of agricultural products

- correlation to exchange-traded assets
- partial hedging possible
- basis risk remains
- incompleteness

Questions:

- Fair prices $=$?
- Explicit cross hedging strategies $=$?
- Dynamic versus static risk?

Questions:

- Fair prices = ?
- Explicit cross hedging strategies = ?
- Dynamic versus static risk?

Answers:

- Carmona, Davis, Henderson, Hobson, Imkeller, Monoyios, Musiela, Popier, Zariphopoulou, ...

1. Example: Weather derivatives based on Heating degree days

- common underlying of weather derivatives
- T_i = average of the maximum and the minimum temperature on day i at a specific location
- $\mathrm{HDD}_i = \max\left(0; 18 - T_i\right)$

Cumulative heating degree days

$$\mathrm{cHDD}_t = \sum_{i=0}^{90} \mathrm{HDD}_{t-i}$$

Derivatives:

- Option: $(K - \mathrm{cHDD})^+$
- Swap: $b(\mathrm{cHDD} - K)$

How to model the dynamics

- cHDD can be modeled as a *geometric Brownian motion*

$$dR_t = \mu(t)R_t dt + \nu(t)R_t dW_t^1$$

(moving average)

- correlated asset

$$dS_t = S_t[\alpha(t, R_t)dt + \beta_1(t, R_t)dW_t^1 + \beta_2(t, R_t)dW_t^2]$$

2. Example: Crack Spreads

- simultaneous purchase of crude oil (crude) against sale of refined petroleum
- risk management tool for oil refiners

Aim: Optimal hedging of a *kerosene crack spread option*:

$$\Big(K - (R_T^{ker} - S_T^{crude})\Big)^+$$

⇒ kerosene is correlated to heating oil (heat) futures

price dynamics:

$$\begin{aligned} dR_t^{ker} &= R_t^{ker}\left(b_1 dt + \gamma_2 dW_t^1 + \gamma_3 dW_t^2 + \gamma_4 dW_t^3\right) \\ dS_t^{heat} &= S_t^{heat}\left(b_2 dt + \beta_1 dW_t^1 + \beta_2 dW_t^2,\right) \\ dS_t^{crude} &= S_t^{crude}\left(b_3 dt + \gamma_1 dW_t^1\right) \end{aligned}$$

General aims:

- dynamics of
  - (a) utility indifference prices = ?
  - (b) marginal utility prices = ?
- Explicit optimal cross hedging strategy:

$$\#\ \text{shares} \ = \frac{\partial \text{ ind. price}}{\partial\ \textit{underlying}} \times \text{correlation coeff.}$$

▶ generalisation of the Δ-hedge

The general model

Index process

$$dR_t = b(t, R_t)dt + \rho(t, R_t)dW_t,$$

R is a Markov process: Conditioned on $R_t = r$ the index solves

$$R_s^{t,r} = r + \int_t^s b(u, R_u^{t,r})du + \int_t^s \rho(u, R_u^{t,r})dW_u, \quad s \in [t, T]. \qquad \text{(I)}$$

Derivative $= F(R_T)$; with

$$F : \mathbb{R}^m \to \mathbb{R} \text{ bounded}$$

Correlated financial market with k risky assets with price process:

$$dS_t^i = S_t^i(\alpha_i(t, R_t)dt + \beta_i(t, R_t)dW_t), \quad i = 1, \dots, k, \tag{II}$$

where $\alpha : [0, T] \times \mathbb{R}^m \to \mathbb{R}^k$ and $\beta : [0, T] \times \mathbb{R}^m \to \mathbb{R}^{k \times d}$.

$W =$ the $d-$dimensional Brownian motion driving R

$$\left.\begin{array}{c} \text{investment strategy } \lambda \\ \text{gain process} \end{array}\right\} = \text{standard}$$

Utility function:

$$U(x) = -e^{-\eta x} \qquad (0 < \eta = \text{ risk aversion})$$

maximal expected utility *with* derivative:

$$V^F(t,v,r) = \sup\{EU(v + G_T^{\lambda,t,r} + F(R_T^{t,r})) : \lambda \text{ adm. }\}$$

$\widehat{\pi}$ is the optimal strategy:

$$EU(v + G_T^{\widehat{\pi},t,r} + F(R_T^{t,r})) = V^F(v,t,r)$$

$\widehat{\pi}$ = optimal investment + hedging

Isolate the hedging part:

maximal expected utility *without* derivative:

$$V^0(t, v, r) = \sup\{EU(v + G_T^{\lambda,t,r}) : \lambda \text{ adm. }\}$$

π is the optimal strategy:

$$EU(v + G_T^{\pi,t,r}) = V^0(v, t, r)$$

derivative hedge

$$\Delta = \widehat{\pi} - \pi$$

Remark 1 $\Delta =$ *the best hedging strategy of* $F(R_T)$*.*

Aim: Explicit representation of Δ!

Stochastic control problem

$$\pi =? \qquad\qquad \widehat{\pi} =?$$

Different approaches:

- HJB partial differential equation
- BSDEs

What is a BSDE?

Parameters:

- ξ r.v. $\mathcal{F}_T$-measurable
- $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ predictable mapping

A BSDE with *terminal condition* ξ and *generator* g is an equation of the type

$$Y_t = \xi - \int_t^T Z_s dW_s + \int_t^T g(s, Y_s, Z_s) ds. \tag{III}$$

A solution is a *pair* of predictable processes (Y, Z) such that (III) makes sense.

The BSDE we need:

Let

- $\theta(t,r) = \beta^*(t,r)(\beta(t,r)\beta^*(t,r))^{-1}\alpha(t,r)$
- $C(t,r) = \{x\beta(t,r) : x \in \mathbb{R}^k\}$
- dist$(z, C(t,r)) = \min\{|z-c| : c \in C(t,r)\}$

Let

$$\begin{aligned} &f : [0,T] \times \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}, \\ &(t,r,z) \mapsto z\theta(t,r) + \frac{1}{2\eta}|\theta(t,r)|^2 - \frac{\eta}{2}\mathrm{dist}^2(z + \frac{1}{\eta}\theta(t,r), C(t,r)), \end{aligned}$$

Note that f grows quadratically

$$|f(t,r,z)| \leq c(1+|z|^2) \quad \text{a.s.}$$

Let $(\widehat{Y}^{t,r}, \widehat{Z}^{t,r}) \in \mathcal{S}^\infty(\mathbb{R}) \otimes \mathcal{H}^2(\mathbb{R}^d)$ be the solution of the quadratic BSDE

$$\widehat{Y}_s^{t,r} = F(R_T^{t,r}) - \int_s^T \widehat{Z}_u^{t,r} dW_u - \int_s^T f(u, R_u^{t,r}, \widehat{Z}_u^{t,r}) du, \quad s \in [t, T],$$

Lemma 1 *(Hu, Imkeller, Müller 2005)*

$$V^F(v, t, r) = -e^{-\eta(v - \widehat{Y}_t^{t,r})}$$

and

$$\widehat{\pi}_s \beta(s, R_s^{t,r}) = \Pi_{C(t,r)}[\widehat{Z}_s^{t,r} + \frac{1}{\eta}\theta(s, R_s^{t,r})], \quad s \in [t, T],$$

where $\Pi_{C(t,r)} =$ *projection operator onto* $C(t, r)$*.*

Similarly: V^0 and π via $(Y^{t,r}, Z^{t,r})$ = the solution of

$$Y_s^{t,r} = 0 - \int_s^T Z_u^{t,r} dW_u - \int_s^T f(u, R_u^{t,r}, Z_u^{t,r}) du, \quad s \in [t,T],$$

Theorem 1 $p(t,r) := Y_t^{t,r} - \widehat{Y}_t^{t,r}$ *is the indifference price, i.e.*

$$V^F(t, v - p(t,r), r) = V^0(t, v, r).$$

And

$$\Delta_s \beta(s, R_s^{t,r}) = \Pi_{C(t,r)}[\widehat{Z}_s^{t,r} - Z_s^{t,r}]$$

So what?

Properties of the BSDE

Theorem 2 *(***Markov property***)*
There exist mb. deterministic functions $\widehat{u}(t,x)$ *and* $\widehat{v}(t,x)$ *s.th.*

$$\widehat{Y}^{t,r}_s \quad = \quad \widehat{u}(s,R^{t,r}_s) \qquad and \qquad \widehat{Z}^{t,r}_s = \widehat{v}(s,R^{t,r}_s)\rho(s,R^{t,r}_s).$$

Corollary 1 *Since* $p(t,r) := Y^{t,r}_t - \widehat{Y}^{t,r}_t$*, the indifference price at time* t

- *depends only on* R_t*!*
- *And not on* S_t*!*

Theorem 3 (**Moment estimates**) *(A., Imkeller, Reis 07)*
Let f be differentiable in r; f and $\nabla_r f$ are Lipschitz, ...
Then for $p > 1$

$$E \sup_{s\in[t,T]} |\widehat{Y}_s^{t,r} - \widehat{Y}_s^{t,r'}|^{2p} \leq C|r-r'|^{2p}$$

and

$$E\left[\left(\int_t^T |\widehat{Z}_s^{t,r} - \widehat{Z}_s^{t,r'}|^2 ds\right)^p\right] \leq C|r-r'|^{2p}$$

Theorem 4 (**Differentiability**) *(A., Imkeller, Reis 07)*
There exists a version of $(\widehat{Y}_s^{t,r}, \widehat{Z}_s^{t,r})$ *such that a.s.*

(1) $\widehat{Y}_s^{t,r}$ *is continuous in* s *and cont. differentiable in* r *(classical sense)*

(2) $\widehat{Z}^{t,r}$ *is differentiable in a weak sense (norm topology)*

(3) $(\nabla_r \widehat{Y}^{t,r}, \nabla_r \widehat{Z}^{t,r})$ *solves the BSDE*

$$\begin{aligned} \nabla_r \widehat{Y}_t^r \quad = \quad & \nabla_r F(R_s^{t,r}) \nabla_r R_s^{t,r} - \int_t^T \nabla_r \widehat{Z}_s^{t,r} dW_s \\ & + \int_t^T \Big[\nabla_r f(s, R_s^{t,r}, \widehat{Z}_s^{t,r}) \nabla_r R_s^{t,r} \\ & \qquad + \nabla_z f(s, R_s^{t,r}, \widehat{Z}_s^{t,r}) \nabla_r \widehat{Z}_s^{t,r} \Big] ds. \end{aligned} \tag{IV}$$

Remark: random Lipschitz constant: $\widehat{Z}_s^{t,r}$

Theorem 5 (Malliavin differentiability) *(A., Imkeller, Reis 07)*

$$D_\vartheta \widehat{Y}_s^{t,r} = \nabla_r \widehat{u}(s, R_s^{t,r}) D_\vartheta R_s^{t,r}$$

and

$$\widehat{Z}_s^{t,r} = D_s \widehat{Y}_s^{t,r} = \nabla_r \widehat{u}(s, R_s^{t,r}) \rho(s, R_s^{t,r})$$

Main economic results

Properties of the BSDEs $\Longrightarrow$

Theorem 6 *The indifference price* $p(t,r) = Y_t^{t,r} - \widehat{Y}_t^{t,r}$ *is differentiable in* r*.*

Theorem 7 *The derivative hedge* Δ *at time* t *depends only on* R_t*, and*

$$\begin{aligned}\Delta(t,r)\beta(t,r) &= \Pi_{C(t,r)}[\widehat{Z}_t^{t,r} - Z_t^{t,r}] \\ &= \Pi_{C(t,r)}[\nabla_r(\widehat{Y}_t^{t,r} - Y_t^{t,r})\rho(s,R_s^{t,r})] \\ &= -\Pi_{C(t,r)}[\nabla_r p(t,r)\rho(t,r)].\end{aligned}$$

Remark:

- *complete* case: $\Delta =$ *'delta hedge'*
- where is the *risk aversion* η?

Example 1:

- cHDD

$$dR_t = \mu(t)R_t dt + \nu(t)R_t dW_t^1$$

- correlated asset

$$dS_t = S_t[\alpha(t, R_t)dt + \beta_1(t, R_t)dW_t^1 + \beta_2(t, R_t)dW_t^2]$$

Then

$$\Delta(t, r) = -\nu(t)\frac{\partial p(t, r)}{\partial r}\frac{\beta_1(t, r)}{(\beta_1^2 + \beta_2^2)(t, r)}.$$

Example 2: Kerosene crack spread

$$\begin{array}{rcl} dR_t^{ker} & = & R_t^{ker}\left(b_1 dt + \gamma_2 dW_t^1 + \gamma_3 dW_t^2 + \gamma_4 dW_t^3\right) \\ dS_t^{heat} & = & S_t^{heat}\left(b_2 dt + \beta_1 dW_t^1 + \beta_2 dW_t^2,\right) \\ dS_t^{crude} & = & S_t^{crude}\left(b_3 dt + \gamma_1 dW_t^1\right) \end{array}$$

Then

$$\Delta(t,r) = \Big(-\frac{\partial}{\partial(crude)} p(t,r) + \big(\frac{\beta_1\gamma_3}{\gamma_1\beta_2} - \frac{\gamma_2}{\gamma_1}\big)\frac{\partial}{\partial(ker)} p(t,r) \qquad -\frac{\gamma_3}{\beta_2}\frac{\partial}{\partial(ker)} p(t,r) \Big).$$

Pricing by marginal utility

Concerns with indifference prices:

- Ind. prices are non-linear:

$$\text{ind. price of } 2 \times F(R_T) \;\neq\; 2 \times \;\text{ind. price of } (F(R_T))$$

$\Longrightarrow$ pricing by marginal utility

Definition 1

$$\sup_{\lambda}\{EU(v + G_T^{\lambda,t,r} + qF(R_T^{t,r}) - p(t,r,q))\} = \sup_{\lambda}\{EU(v + G_T^{\lambda,t,r})\}$$

Pricing formula:

$$MUP = \frac{\partial p(t,r,q)}{\partial q}\Big|_{q=0}$$

Recall: $p(t,r,q) = Y_t^{t,r} - \widehat{Y}_t^{t,r,q}$, where

$$\widehat{Y}_s^{t,r,q} = qF(R_T^{t,r}) - \int_s^T \widehat{Z}_u^{t,r,q} dW_u - \int_s^T f(u, R_u^{t,r}, \widehat{Z}_u^{t,r,q}) du, \quad s \in [t,T],$$

The derivative wrt q:

$$\frac{\partial}{\partial q}\widehat{Y}_s^{t,r,q} = F(R_T^{t,r}) - \int_s^T \frac{\partial}{\partial q}\widehat{Z}_u^{t,r,q} dW_u - \int_s^T \frac{\partial}{\partial z} f(u, R_u^{t,r}, \widehat{Z}_u^{t,r,q}) \frac{\partial}{\partial q}\widehat{Z}_u^{t,r,q} du,$$

Put $q = 0 \Longrightarrow$

$$U_s^{t,r} = F(R_T^{t,r}) - \int_s^T V_u dW_u - \int_s^T \frac{\partial}{\partial z} f(u, R_u^{t,r}, Z_u^{t,r}) V_u du \qquad \text{(V)}$$

Theorem 8

$$MUP := \frac{\partial p(t,r,q)}{\partial q}\Big|_{q=0} = U_t^{t,r}$$

Remark: (V) is a BSDE with random Lipschitz condition

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Thanks for your attention!