Investment performance measurement, risk tolerance and optimal portfolio choice

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Investment performance measurement

Investment performance measurement

Traditional framework

A deterministic utility datum $u_T(x)$ is assigned at the end of a fixed investment horizon

$$U_T(x) = u_T(x)$$

Optimality

$$V_t(x) = \sup_{\pi} E_{\mathbb{P}}(u_T(X_T^{\pi})|\mathcal{F}_t; X_t^{\pi} = x)$$

$$V_t(x) = \sup_{\pi} E_{\mathbb{P}}(V_s(X_s^{\pi})|\mathcal{F}_t; X_t^{\pi} = x) \qquad \text{(DPP)}$$

$$V_t(x) = E_{\mathbb{P}}(V_s(X_s^{\pi^*})|\mathcal{F}_t; X_t^{\pi^*} = x), \qquad 0 \leq t < t$$

Optimality beyond T ?

Investment performance process

$U_t(x)$ is an $\mathcal{F}_t$-adapted process, $t \geq 0$

- As a function of x, U is increasing and concave
- For each self-financing strategy, represented by π, the associated (discounted) wealth X_t satisfies

$$E_{\mathbb{P}}(U_t(X_t^{\pi}) \mid \mathcal{F}_s) \leq U_s(X_s^{\pi}) \qquad 0 \leq s \leq t$$

- There exists a self-financing strategy, represented by π^*, for which the associated (discounted) wealth $X_t^{\pi^*}$ satisfies

$$E_{\mathbb{P}}(U_t(X_t^{\pi^*}) \mid \mathcal{F}_s) = U_s(X_s^{\pi^*}) \qquad 0 \leq s \leq t$$

Alternative framework

A deterministic datum $u_0(x)$ is assigned at the beginning of the trading horizon, $t = 0$

$$U_0(x) = u_0(x)$$

Forward in time generation of optimal performance

$$U_s(X_s^{\pi^*}) = E_{\mathbb{P}}(U_t(X_t^{\pi^*})|\mathcal{F}_s) \qquad 0 \leq s \leq t$$

- Performance can be defined for all trading horizons
- Difficulties due to the "inverse in time" nature of the problem

References

- Indifference valuation in binomial models (with M. Musiela)
- Investments and forward utilities (with M. Musiela)
- Investment and valuation under backward and forward dynamic exponential utilities in a stochastic factor model (with M. Musiela)
- Optimal asset allocation under forward exponential criteria (with M. Musiela)
- Horizon-independent risk measures (with G. Zitkovic)

Motivational examples

An incomplete multiperiod binomial example

Exponential datum

- Traded security: S_t, $t = 0, 1, ...$

  $\xi_{t+1} = \frac{S_{t+1}}{S_t}$, $\xi_{t+1} = \xi_{t+1}^d, \xi_{t+1}^u$ with $0 < \xi_{t+1}^d < 1 < \xi_{t+1}^u$

  Second traded asset is riskless yielding zero interest rate

- Stochastic factor: Y_t, $t = 0, 1, ...$

  $\eta_{t+1} = \frac{Y_{t+1}}{Y_t}$, $\eta_{t+1} = \eta_{t+1}^d, \eta_{t+1}^u$ with $\eta_t^d < \eta_t^u$

- Probability space $(\Omega, (\mathcal{F}_t), \mathbb{P})$

  $\{S_t, Y_t : t = 0, 1, ...\}$: a two-dimensional stochastic process

- State wealth process: X_t, $t = s+1, s+2, \ldots, \ldots$

  α_i : the number of shares of the traded security held in this portfolio over the time period $[i-1, i]$

$$X_t = X_s + \sum_{i=s+1}^{t} \alpha_i \triangle S_i$$

- Forward exponential performance

$$\begin{cases} U_s(X_s^{\alpha^*}) = E_{\mathbb{P}}(U_t(X_t^{\alpha^*})|\mathcal{F}_s) \\ U_0(x) = -e^{-\gamma x}\,, \qquad \gamma > 0 \end{cases}$$

- A forward performance process

$$U_t(x) = \begin{cases} -e^{-\gamma x} & \text{if} \quad t = 0 \\ -e^{-\gamma x + \sum_{i=1}^{t} h_i} & \text{if} \quad t \geq 1 \end{cases}$$

- Auxiliary quantities

$$h_i = q_i \log \frac{q_i}{\mathbb{P}\left(A_i \,|\mathcal{F}_{i-1}\right)} + (1 - q_i) \log \frac{1 - q_i}{1 - \mathbb{P}\left(A_i \,|\mathcal{F}_{i-1}\right)}$$

with

$$A_i = \{\xi_i = \xi_i^u\} \quad \text{and} \quad q_i = \mathbb{Q}\left(A_i \,|\mathcal{F}_{i-1}\right)$$

for $i = 0, 1, ..$ and $\mathbb{Q}$ being the minimal martingale measure

Important insights

The forward performance process

$$U_t(x) = -e^{-\gamma x + \sum_{i=1}^{t} h_i}$$

is of the form

$$U_t(x) = u(x, A_t)$$

where $u(x,t)$ is the deterministic function

$$u(x,t) = -e^{-\gamma x + \frac{1}{2}t}$$

and A_t corresponds to a time change depending on the "market input"

$$A_t = 2\sum_{i=1}^{t} h_i$$

A continuous-time example

Arbitrary datum

- Investment opportunities

  Riskless bond : $r = 0$

  Risky security : $dS_t = \sigma_t S_t(\lambda_t dt + dW_t)$

- Datum at $t = 0$: $u_0(x)$ concave, increasing

- Wealth process

$$\begin{cases} dX_t = \sigma_t \pi_t(\lambda_t dt + dW_t) \\ X_0 = x \end{cases}$$

- Market input : λ_t, A_t

$$\begin{cases} dA_t = \lambda_t^2 dt \\ A_0 = 0 \end{cases}$$

- Building the martingale $U_t(X_t^{\pi^*})$

Assume that we can construct $U_t(x)$ via

$$U_t(X_t^{\pi^*}) = u(X_t^{\pi^*}, A_t)$$

where $u(x,t)$ is the differential utility input and A_t the stochastic market input

$$dU_t(X_t^{\pi}) = u_x(X_t, A_t)\sigma_t \pi_t \, dW_t$$

$$+\underbrace{(u_t(X_t^{\pi}, A_t)\lambda_t^2 + u_x(X_t^{\pi}, A_t)\sigma_t \pi_t \lambda_t + \tfrac{1}{2}u_{xx}(X_t^{\pi}, A_t)\sigma_t^2 \pi_t^2)dt}$$

$$\| \quad \alpha_t = \lambda_t^{-1}\sigma_t \pi_t$$

$$+\lambda_t^2(u_t(X_t^{\pi}, A_t) + u_x(X_t^{\pi}, A_t)\alpha_t + \tfrac{1}{2}u_{xx}(X_t^{\pi}, A_t)\alpha_t^2)dt$$

- Differential utility input condition

$$\begin{cases} u_t \; u_{xx} = \frac{1}{2} u_x^2 \\ \\ u(x,0) = u_0(x) \end{cases}$$

- The optimal allocations in stock, π_t^*, and in bond, $\pi_t^{0,*}$,

$$\begin{cases} \pi_t^* = -\sigma_t^{-1} \lambda_t \dfrac{u_x(X_t^{\pi^*}, A_t)}{u_{xx}(X_t^{\pi^*}, A_t)} = \sigma_t^{-1} \lambda_t R_t \\ \\ \pi_t^{0,*} = X_t^{\pi^*} - \sigma_t^{-1} \lambda_t R_t \end{cases}$$

$$R_t = r(X_t^{\pi^*}, A_t) \quad ; \qquad r(x,t) = -\frac{u_x(x,t)}{u_{xx}(x,t)}$$

The local risk tolerance $r(x,t)$ and the risk tolerance process R_t emerge as important quantities

Forward performance measurement

time t_1, information $\mathcal{F}_{t_1}$

asset returns

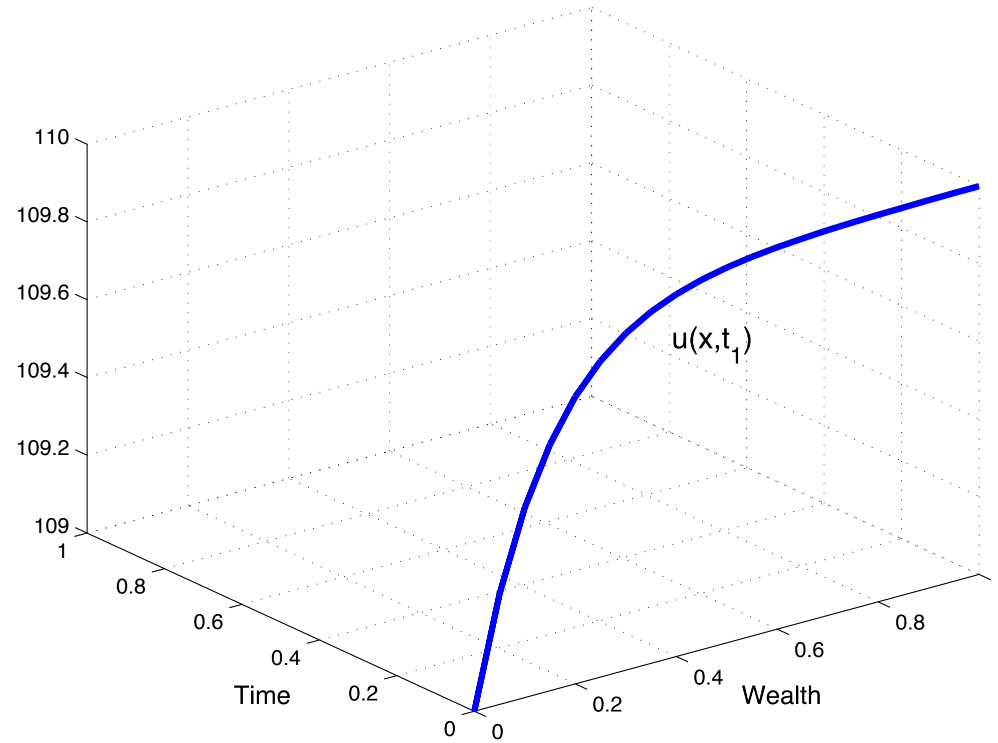
constraints

market view
away from equilibrium

benchmark numeraire

calendar time subordination

$\longrightarrow$



$$MI(t_1) \dashrightarrow \quad + \quad \dashleftarrow u(x, t_1)$$

$$\Downarrow$$

$$\boxed{U_{t_1}(x; MI) \in \mathcal{F}_{t_1} \qquad \pi_{t_1}(x; MI) \in \mathcal{F}_{t_1}}$$

Forward performance measurement

time t_2, information $\mathcal{F}_{t_2}$

asset returns

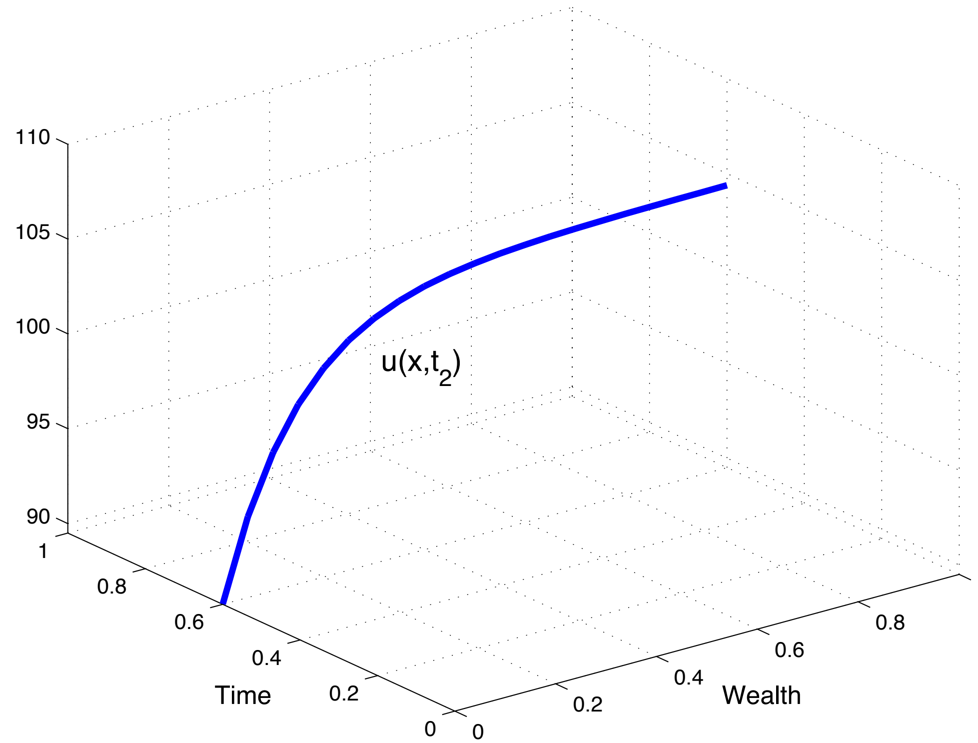
constraints

market view
away from equilibrium

benchmark numeraire

calendar time subordination

$\longrightarrow$



$$MI(t_2) \dashrightarrow + \dashleftarrow u(x, t_2)$$

$$\Downarrow$$

$U_{t_2}(x; MI) \in \mathcal{F}_{t_2}$	$\pi_{t_2}(x; MI) \in \mathcal{F}_{t_2}$

Forward performance measurement

time t_3, information $\mathcal{F}_{t_3}$

asset returns

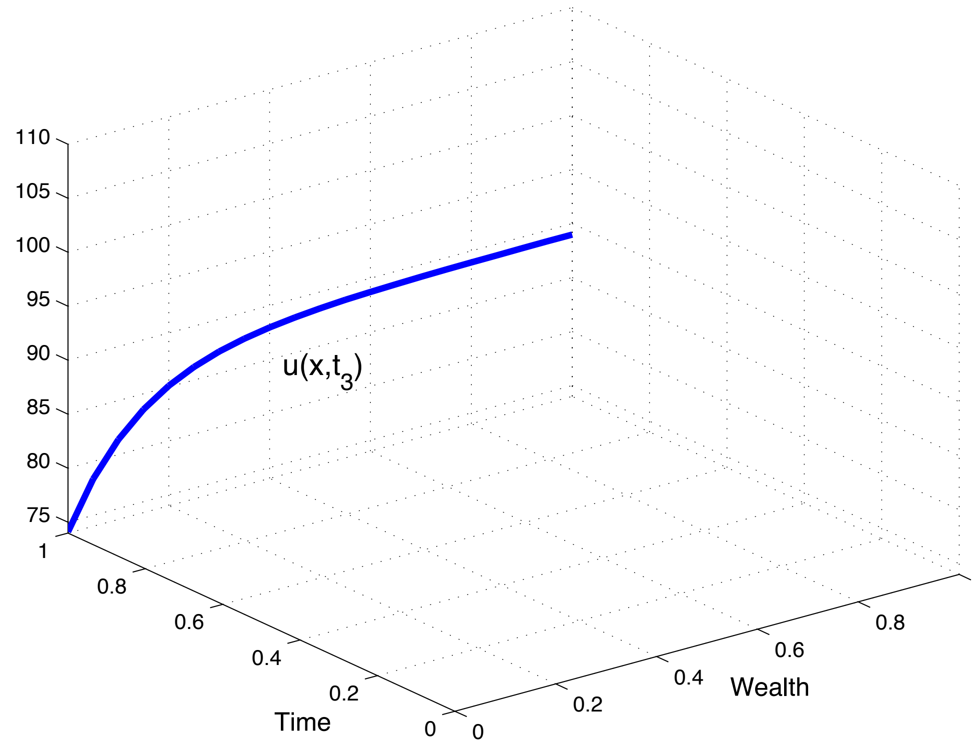
constraints

market view
away from equilibrium

benchmark   numeraire

calendar time   subordination

$\longrightarrow$



$$MI(t_3) \dashrightarrow \quad + \quad \dashleftarrow u(x, t_3)$$

$$\Downarrow$$

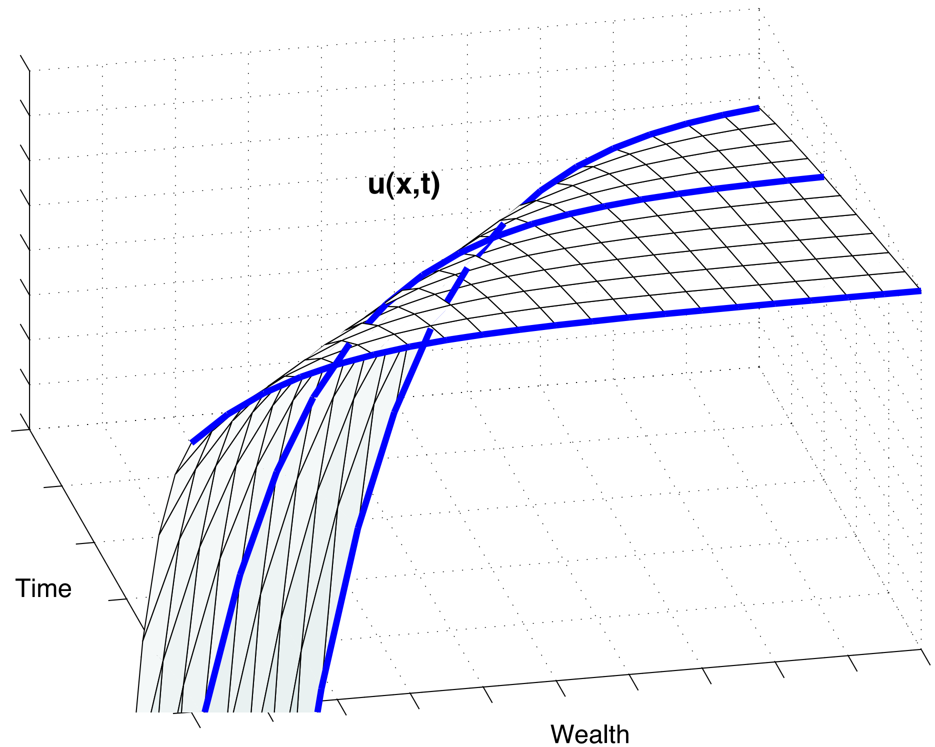
$$\boxed{U_{t_3}(x; MI) \in \mathcal{F}_{t_3} \qquad \pi_{t_3}(x; MI) \in \mathcal{F}_{t_3}}$$

Forward performance measurement

time t, information $\mathcal{F}_t$

asset returns additional market input

$\longrightarrow$



$$MI(t) \quad \dashrightarrow \quad + \quad \dashleftarrow \quad u(x,t)$$

$$\Downarrow$$

$$\boxed{U_t(X_t^*) \in \mathcal{F}_t \qquad \pi_t^*(X_t^*) \in \mathcal{F}_t}$$

Construction of a class of forward performance processes

Creating the martingale that yields the optimal performance

Minimal model assumptions

Key idea

Stochastic input
Market

Differential input
Individual

Maximal performance — Optimal allocation

Differential input

Performance surface

A model independent differential constraint on

impatience, risk aversion and monotonicity

- Initial datum

$$u_0(x) = u(x, 0)$$

- Fully non-linear pde

$$\begin{cases} u_t \ u_{xx} = \frac{1}{2} u_x^2 \\ u(x,0) = u_0(x) \end{cases}$$

Transport equation

The u-equation can be alternatively viewed as a transport equation with slope of its characteristics equal to (half of) the risk tolerance

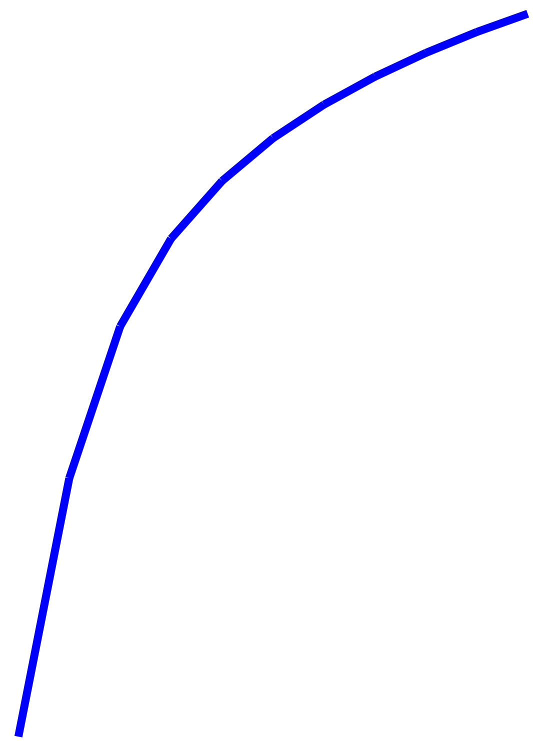
$$r(x,t) = -\frac{u_x(x,t)}{u_{xx}(x,t)}$$

$$\begin{cases} u_t + \frac{1}{2} r(x,t) u_x = 0 \\ u(x,0) = u_0(x) \end{cases}$$

Characteristic curves: $$\frac{dx(t)}{dt} = \frac{1}{2} r(x(t),t)$$

Construction of differential input $u(x,t)$ using characteristics

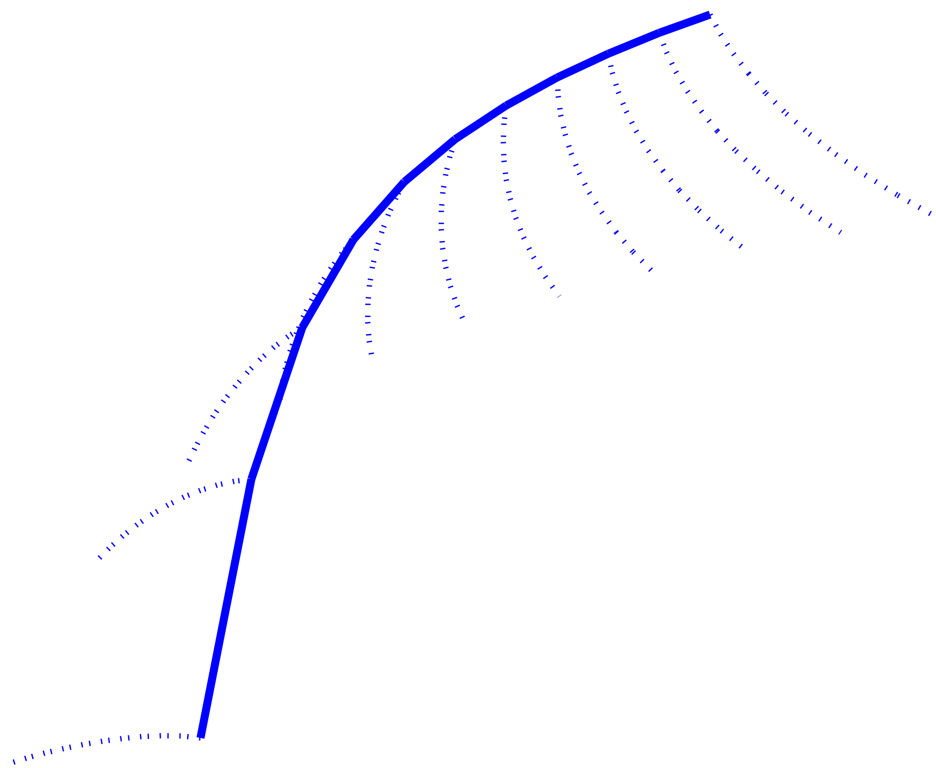
$$\frac{dx(t)}{dt} = \frac{1}{2} r(x(t), t)$$



Performance datum $u(x,0)$

Construction of characteristics

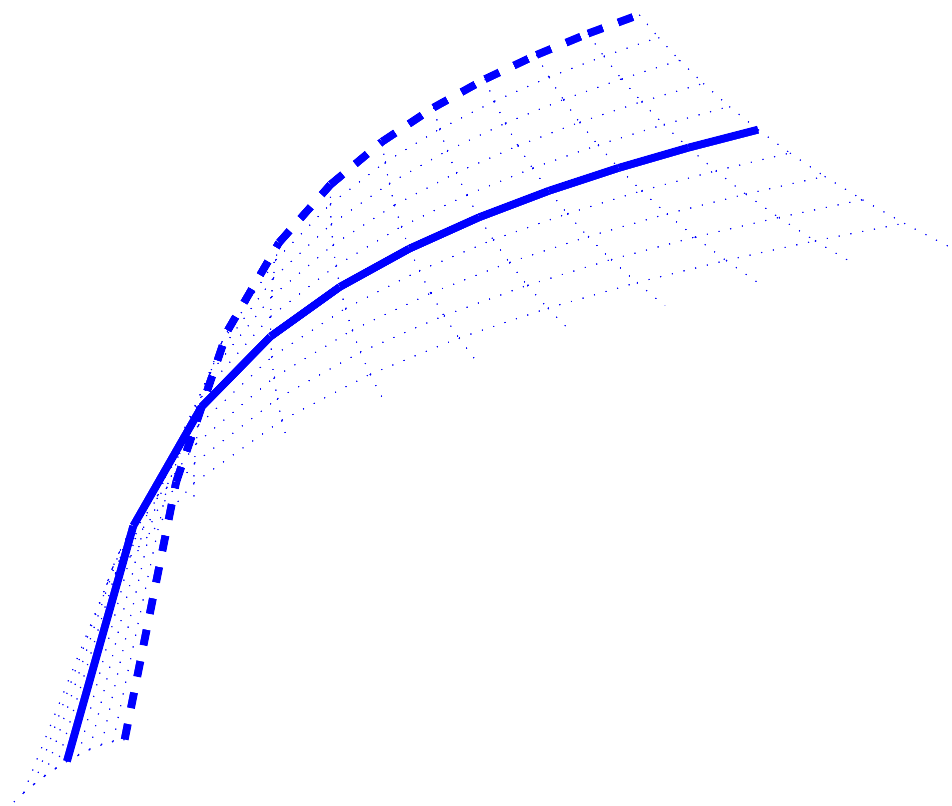
$$\frac{dx(t)}{dt} = \frac{1}{2} r(x(t), t)$$



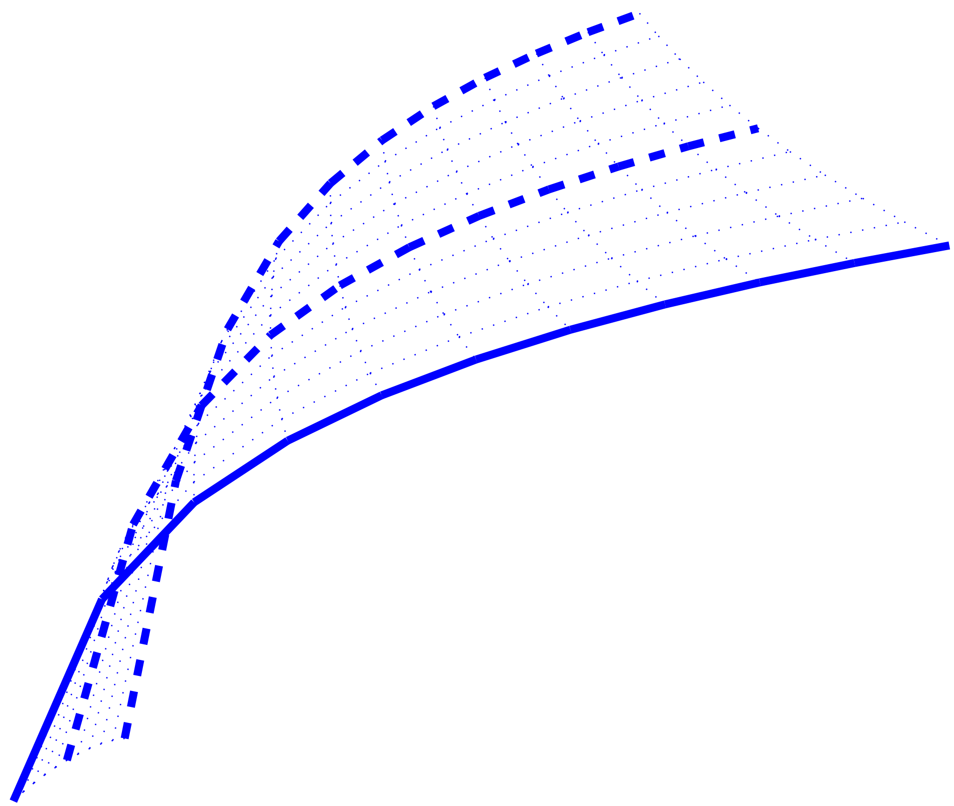
Performance datum $u(x, 0)$

Characteristic curves

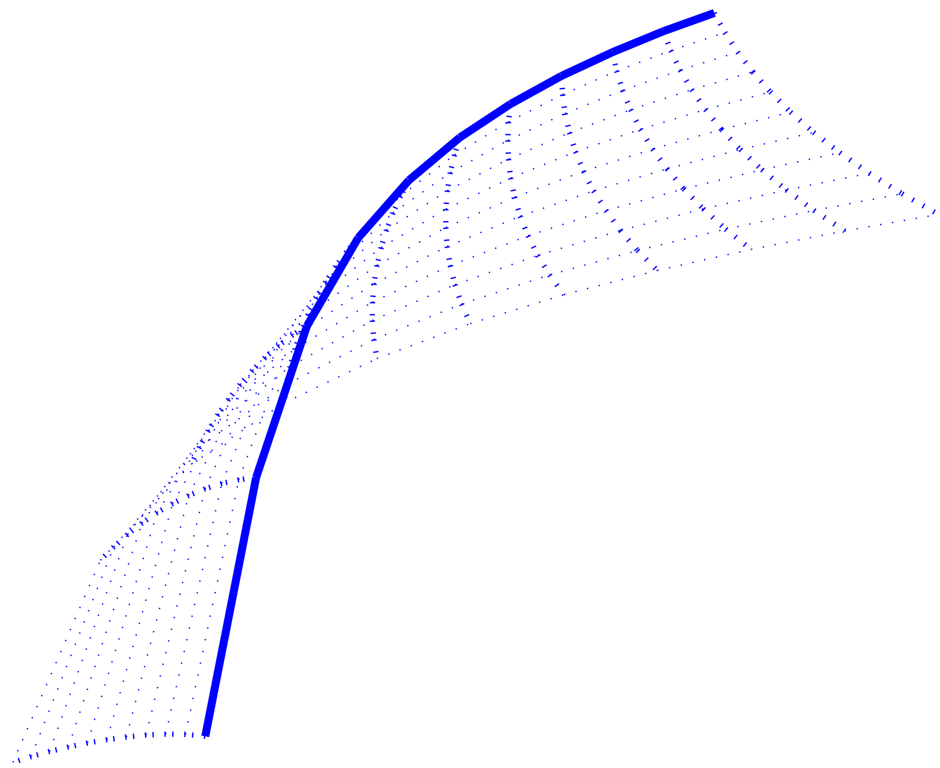
Propagation of performance datum along characteristics



Propagation of performance datum along characteristics



Differential input $u(x,t)$



Two related pdes

- Fast diffusion equation for risk tolerance

$$\begin{cases} r_t + \frac{1}{2} r^2 r_{xx} = 0 \\ r(x,0) = r_0(x) \end{cases} \qquad \text{(FDE)}$$

Conductivity : r^2

- Porous medium equation for risk aversion

$$\gamma(x,t) = \frac{1}{r(x,t)}$$

$$\begin{cases} \gamma_t = \frac{1}{2}\left(\frac{1}{\gamma}\right)_{xx} \\ \gamma(x,0) = \frac{1}{r_0(x)} \end{cases} \qquad \text{(PME)}$$

Pressure : r^2 and (PME) exponent: $m = -1$

Difficulties

- Differential input equation: $u_t\ u_{xx} = \frac{1}{2}u_x^2$

  Inverse problem and fully nonlinear

- Transport equation: $u_t + \frac{1}{2}r(x,t)u_x = 0$

  Shocks, solutions past singularities

- Fast diffusion equation: $r_t + \frac{1}{2}r^2 r_{xx} = 0$

  Inverse problem and backward parabolic, solutions might not exist, locally integrable data might not produce locally bounded slns in finite time

- Porous medium equation: $\gamma_t = \frac{1}{2}(\frac{1}{\gamma})_{xx}$

  Majority of results for (PME), $\gamma_t = (\gamma^m)_{xx}$, are for $m > 1$, partial results for $-1 < m < 0$

A rich class of risk tolerance differential inputs

- Addititively separable risk tolerance

$$r^2(x,t;\alpha,\beta) = \alpha x^2 + \beta e^{-\alpha t}$$

$$r(x,t;\alpha,\beta) = \sqrt{\alpha x^2 + \beta e^{-\alpha t}} \qquad \alpha, \beta > 0$$

$$r(x,t;\alpha,\beta) = \sqrt{\alpha (x - x_0)^2 + \beta e^{-\alpha t}}$$

(Very) special cases

$$r(x,t;0,\beta) = \sqrt{\beta} \quad \longrightarrow \quad u(x,t) = -e^{-\frac{x}{\sqrt{\beta}} + \frac{t}{2}}$$

$$r(x,t;1,0) = |x| \quad \longrightarrow \quad u(x,t) = \log x - \frac{t}{2}$$

$$r(x,t;\alpha,0) = \sqrt{\alpha}\ |x| \quad \longrightarrow \quad u(x,t) = \frac{1}{\gamma} x^\gamma e^{-\frac{\gamma}{2(1-\gamma)}t}, \qquad \gamma = \frac{\sqrt{\alpha}-1}{\sqrt{\alpha}}$$

Multiplicatively separable risk tolerance

$$r(x,t;\alpha,\beta) = m(x;\alpha)n(t;\beta)$$

Example

$$m(x;\alpha) = \varphi(\Phi^{-1}(x;\alpha)) \qquad n(t;\beta) = \frac{1}{\sqrt{t+\beta}}\,, \qquad \beta > 0$$

$$\Phi(x;\alpha) = \int_\alpha^x e^{z^2/2}\,dz \qquad \varphi = \Phi'$$

$$r(x,t;\alpha,\beta) = \varphi(\Phi^{-1}(x;\alpha))$$

(Very) special cases

$$m(x;\alpha) = \alpha,\ n(t;\beta) = 1 \quad \longrightarrow \quad u(x,t) = -e^{-\frac{x}{\alpha}+\frac{t}{2}}$$

$$m(x;\alpha) = x,\ n(t;\beta) = 1 \quad \longrightarrow \quad u(x,t) = \log x - \frac{t}{2}$$

$$m(x;\alpha) = \alpha x,\ n(t;\beta) = 1 \ \longrightarrow \ u(x,t) = \frac{1}{\gamma}x^\gamma e^{-\frac{\gamma}{2(1-\gamma)}t}, \quad \gamma = \frac{\alpha-1}{\alpha}$$

Summary on differential input

- Key state variables: wealth and risk tolerance

- Risk tolerance solves a fast diffusion equation posed inversely in time

$$\begin{cases} r_t + \frac{1}{2} r^2 r_{xx} = 0 \\ r(x,0) = -\frac{u_0'(x)}{u_0''(x)} \end{cases}$$

- Transport equation

$$\begin{cases} u_t + \frac{1}{2} r(x,t) u_x = 0 \\ u(x,0) = u_0(x) \end{cases}$$

Forward performance process constructed by compiling differential input and stochastic market input

Stochastic market input

Investment universe

Riskless and risky securities

- $(\Omega, \mathcal{F}, \mathbb{P})$; $W = (W^1, \ldots, W^d)$ standard Brownian Motion

- Traded securities

$$1 \le i \le k \qquad \begin{cases} dS_t^i = S_t^i(\mu_t^i dt + \sigma_t^i \cdot dW_t) \ , & S_0^i > 0 \\ dB_t = r_t B_t dt \ , & B_0 = 1 \end{cases}$$

$\mu_t, r_t \in \mathbb{R}$, $\sigma_t^i \in \mathbb{R}^d$ bounded and $\mathcal{F}_t$-measurable stochastic processes

- Postulate existence of a $\mathcal{F}_t$-measurable stochastic process $\lambda_t \in \mathbb{R}^d$ satisfying

$$\mu_t - r_t \, \mathbb{1} = \sigma_t^T \lambda_t$$

Investment universe

- Self-financing investment strategies $\pi_t^0, \pi_t^i, \quad i = 1, \ldots, k$

- Present value of this allocation

$$X_t = \sum_{i=0}^{k} \pi_t^i$$

$$dX_t = \sum_{i=0}^{k} \pi_t^i (\mu_t^i - r_t)\, dt + \sum_{i=0}^{k} \pi_t^i \sigma_t^i \cdot dW_t$$

$$= \sigma_t \pi_t \cdot (\lambda_t\, dt + dW_t)$$

$$\pi_t = (\pi_t^1, \ldots, \pi_t^k), \qquad \mu_t - r_t\, \mathbb{1} = \sigma_t^T \lambda_t$$

Market input processes

(σ_t, λ_t) **and** (Y_t, Z_t, A_t)

These $\mathcal{F}_t$-mble processes do **not** depend on the investor's differential input

They reflect and represent, respectively

(λ_t, σ_t) : dynamics of traded securites

Y_t : benchmark
numeraire

Z_t : market view away from market equilibrium
feasibility and trading constraints

A_t : time change

The processes (Y_t, Z_t, A_t)

- Benchmark and/or numeraire

$$\begin{cases} dY_t = Y_t \delta_t \cdot (\lambda_t dt + dW_t) \\ Y_0 = 1 \end{cases}$$

$$\delta_t \in \mathcal{F}_t \;, \qquad \sigma_t \sigma_t^+ \delta_t = \delta_t$$

σ_t^+ : Moore-Penrose matrix inverse

Market input processes

- Market views, feasibility and trading constraints

An exponential martingale Z_t satisfying

$$\begin{cases} dZ_t = Z_t \phi_t \cdot dW_t \\ Z_0 = 1 \ , \quad \phi_t \in \mathcal{F}_t \end{cases}$$

- Time rescaling

A non-decreasing process A_t solving

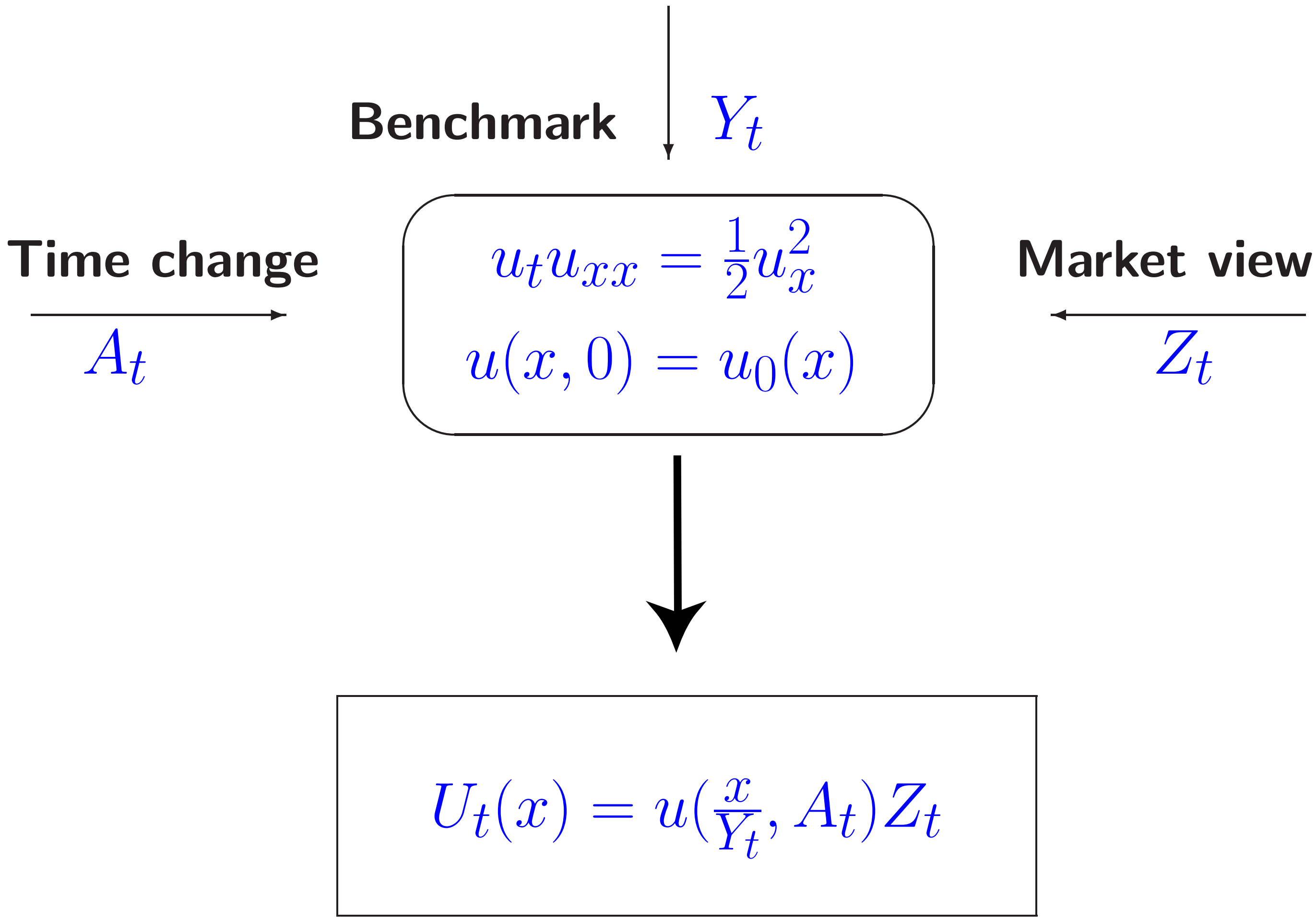
$$\begin{cases} dA_t = |\delta_t - \sigma_t \sigma_t^{+} (\lambda_t + \phi_t)|^2 \, dt \\ A_0 = 0 \end{cases}$$

Forward performance process
Optimal asset allocation

Forward performance process

Stochastic input : (Y_t, Z_t, A_t)

Differential input : $u(x,t)$



Forward performance process

Stochastic market input

$\lambda_t, \ \sigma_t$

$\downarrow$

benchmark, views

subordination

$(Y_t, \ Z_t, \ A_t)$

Differential input

$$x, r_0(x) = -\frac{u_0'(x)}{u_0''(x)}$$

$\downarrow$

$$r_t + \frac{1}{2} r^2 r_{xx} = 0 \quad \text{(FDE)}$$

$$u_t + \frac{1}{2} r u_x = 0 \quad \text{(TE)}$$

$u(x,t)$

$$U_t(x) = u\left(\frac{x}{A_t}, Y_t\right) Z_t$$

Model independent construction!

What is the optimal allocation?

Optimal portfolio processes

$$\pi_t = (\pi_t^0, \pi_t^1, \ldots, \pi_t^k)$$

can be directly and explicitly characterized

along with the construction of the forward performance!

The structure of optimal portfolios

$$dX_t^* = \sigma_t \pi_t^* \cdot (\lambda_t \, dt + dW_t)$$

Stochastic input
Market

$(Y_t,\ Z_t,\ A_t)$
$\lambda_t,\ \sigma_t,\ \delta_t,\ \phi_t$

Differential input
Individual

wealth x
risk tolerance $r(x,t)$

π_t^* is a *linear* combination

of optimal wealth

and risk tolerance

Optimal asset allocation

- Let X_t^* be the optimal wealth, Y_t the benchmark and A_t the time rescaling processes

$$dX_t^* = \sigma_t \pi_t^* \cdot (\lambda_t dt + dW_t)$$
$$dY_t = Y_t \delta_t \cdot (\lambda_t dt + dW_t)$$
$$dA_t = |\sigma_t \sigma_t^+ (\lambda_t + \phi_t) - \delta_t|^2 dt$$

- Define

$$\hat{X}_t^* \triangleq \frac{X_t^*}{Y_t} \qquad \text{and} \qquad \widehat{R}_t^* \triangleq r(\hat{X}_t^*, A_t)$$

Optimal (benchmarked) portfolios

$$\hat{\pi}_t^* \triangleq \frac{1}{Y_t} \pi_t^* = \sigma_t^+ (\lambda_t + \phi_t - \delta_t) \widehat{R}_t^* + \delta_t \hat{X}_t^*$$

Stochastic evolution of wealth-risk tolerance

A system of SDEs at the optimum

$$\widehat{X}_t^* = \frac{X_t^*}{Y_t} \quad \text{and} \quad \widehat{R}_t^* = r(\widehat{X}_t^*, A_t)$$

$$\begin{cases} d\widehat{X}_t^* = \widehat{R}_t^*(\sigma_t \sigma_t^+(\lambda_t + \phi_t) - \delta_t) \cdot ((\lambda_t - \delta_t)\, dt + dW_t) \\ d\widehat{R}_t^* = r_x(\widehat{X}_t^*, A_t) d\widehat{X}_t^* \end{cases}$$

- Separability of wealth dynamics in terms of risk tolerance and market input
- Sensitivity of risk tolerance in terms of its spatial gradient and changes in optimal wealth

Wealth – Risk tolerance

Optimal wealth-risk tolerance $(\widehat{X}_t^*, \widehat{R}_t^*)$ system

of SDEs in original market configuration

$$
\begin{cases}
d\widehat{X}_t^* = \widehat{R}_t^* (\sigma_t \sigma_t^+ (\lambda_t + \phi_t) - \delta_t) \cdot ((\lambda_t - \delta_t)\, dt + dW_t) \\
\\
d\widehat{R}_t^* = r_x(\widehat{X}_t^*, A_t)\, d\widehat{X}_t^*
\end{cases}
$$

$$\downarrow$$

change of measure	change of time
historical $\rightarrow$ benchmarked	Levy's theorem

Wealth – Risk tolerance

Optimal wealth-risk tolerance (x_t^1, x_t^2) system of SDEs

in canonical market configuration

$$x_t^1 = \left(\frac{X_t^*}{Y_t}\right)_{A_t^{(-1)}} \qquad x_t^2 = r\left(\frac{X_t^*}{Y_t}, A_t\right)_{A_t^{(-1)}}$$

$$\langle M_t \rangle = A_t \qquad w_t = M_{A^{(-1)}}$$

$$\downarrow$$

$$\begin{cases} dx_t^1 = x_t^2 \, dw_t \\ dx_t^2 = r_x(x_t^1, t) x_t^2 \, dw_t \\ x_0^1 = \frac{x}{y} \ , \quad x_0^2 = r_x(\frac{x}{y}, 0) \end{cases}$$

Analytic solution of the SDE system

$$\begin{cases} dx_t^1 = x_t^2 \, dw_t \\ dx_t^2 = r_x(x_t^1, t) x_t^2 \, dw_t \end{cases}$$

- Define the budget capacity function $h(x,t)$ via

$$x = \int_{\underline{x}}^{h(x,t)} \frac{du}{r(u,t)} = \int_{\underline{x}}^{h(x,t)} \gamma(u,t) du$$

$\underline{x}$: related to symmetry properties of risk tolerance

Analytic solutions

The budget capacity function h solves the (inverse) heat equation

$$\begin{cases} h_t + \frac{1}{2}h_{xx} - \frac{1}{2}r_x(\underline{x}, t)h_x = 0 \\ h(x, 0) = h_0(x) \ , \quad x = \displaystyle\int_{\underline{x}}^{h_0(x)} \frac{du}{r(u, 0)} \end{cases}$$

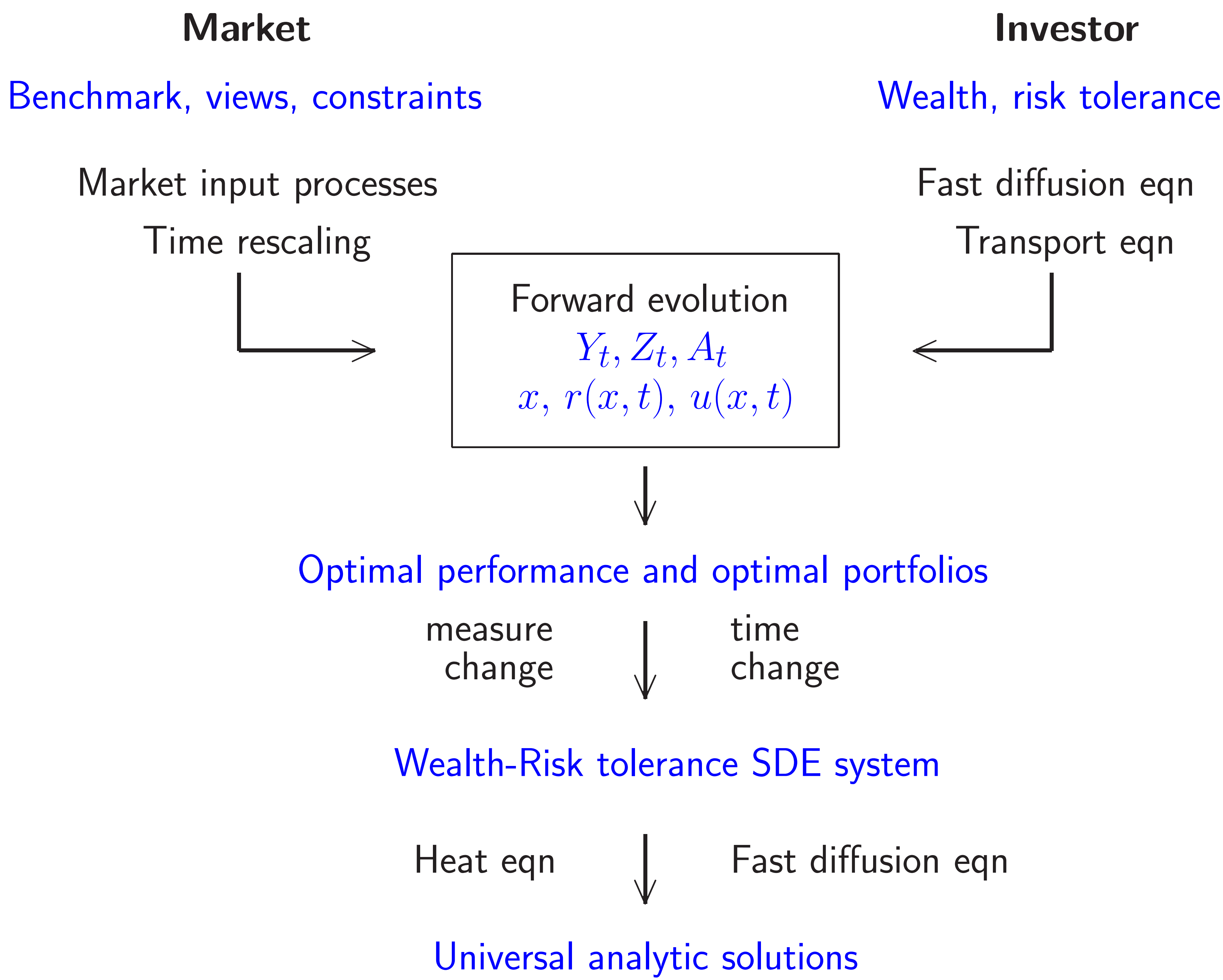
Solution of the SDE system

$$\begin{cases} x_t^1 = h(z_t, t) \\ x_t^2 = h_z(z_t, t) \end{cases}$$

$$z_t = h_0^{-1}(x) - \int_0^t \frac{1}{2} r_x(\underline{x}, s)ds + w_t$$

Using equivalent measure transformations and time change we recover the original pair of optimal (benchmarked) wealth and (benchmark) risk tolerance

Forward performance measurement



Forward performance measurement

Market

Benchmark, views, constraints

Market input processes

Time rescaling

Investor

Wealth, risk tolerance

Fast diffusion eqn

Transport eqn

Forward evolution

Y_t, Z_t, A_t

$x,\ r(x,t),\ u(x,t)$

Optimal performance and optimal portfolios

measure change

time change

Wealth-Risk tolerance SDE system

Heat eqn

Fast diffusion eqn

Universal analytic solutions

An example

Forward exponential performance

Objective: Find an $\mathcal{F}_t-$adapted process $U_t(x)$ such that

$$\begin{cases} U_0(x) = -e^{-x} \\ \\ E_{\mathbb{P}}\left(U_s\left(X_s^{\pi}\right)|\mathcal{F}_t\right) \leq U_t\left(X_t^{\pi}\right) \\ \\ E_{\mathbb{P}}\left(U_s\left(X_s^{\pi^*}\right)|\mathcal{F}_t\right) = U_t\left(X_t^{\pi^*}\right), \quad s \geq t \end{cases}$$

Solution

- Differential input

$$u(x,t) = -e^{-x+\frac{1}{2}t}$$

Forward exponential performance (continued)

- Stochastic market input (Y, Z, A)

$$\begin{cases} dY_t = Y_t \delta_t \cdot (\lambda_t dt + dW_t) \\ Y_0 = y > 0 \ , \end{cases}$$

$$\begin{cases} dZ_t = Z_t \phi_t \cdot dW_t \\ Z_0 = 1 \end{cases}$$

and

$$\begin{cases} dA_t = \left| \delta_t - \sigma_t \sigma_t^{+} (\lambda_t + \phi_t) \right|^2 dt \\ A_0 = 0 \end{cases}$$

Solutions

- Forward performance

$$U_t(x) = -\exp\left(-\frac{x}{Y_t} + \frac{1}{2}\int_0^t |\delta_s - \sigma_s\sigma_s^+(\lambda_s + \phi_s)|^2\, ds - \frac{1}{2}\int_0^t |\phi_s|^2\, ds + \int_0^t \phi_s \cdot dW_t\right)$$

- Feedback portfolio control process

$$\pi_t^* = Y_t\sigma_t^+(\lambda_t + \phi_t - \delta_t) + X_t^*\sigma_t^+\delta_t$$

- Optimal wealth process

$$dX_t^* = \left(Y_t\left(\sigma_t\sigma_t^+\left(\lambda_t + \phi_t\right) - \delta_t\right) + X_t^*\delta_t\right) \cdot \left(\lambda_t dt + dW_t\right)$$

- Optimal performance

$$dU_t\left(X_t^*\right) = U_t\left(X_t^*\right)\left(\sigma_t\sigma_t^+\left(\delta_t - \lambda_t\right) + \left(I - \sigma_t\sigma_t^+\right)\phi_t\right) \cdot dW_t$$

Examples

Case 1: No benchmark and 'no' views $\delta = \phi = 0$

Then, $Y_t = y$, for $t \geq 0$

- Forward performance process

$$U_t(x) = -\exp\left(-\frac{x}{y} + \int_0^t \frac{1}{2}\left|\sigma_s\sigma_s^+\lambda_s\right|^2 ds\right)$$

Note that even in this simple case, the solution is equal to the classical exponential "utility" only at $t = 0$

- Optimal discounted wealth and optimal asset allocation

$$X_t^* = x + \int_0^t y\left(\sigma_s\sigma_s^+\lambda_s\right) \cdot (\lambda_s ds + dW_s)$$

and

$$\pi_t^* = y\sigma_t^+\lambda_t$$

Observe that π^* is independent of the initial wealth x

Case 1: **No benchmark and 'no' views** $\delta = \phi = 0$ (continued)

- Optimal performance

$$U_t\left(X_t^*\right) = -\exp\left(-\frac{x}{y} - \int_0^t \frac{1}{2}\left|\sigma_s\sigma_s^+\lambda_s\right|^2 ds - \int_0^t \sigma_s\sigma_s^+\lambda_s \cdot dW_s\right)$$

- Total amount allocated in the risky assets

$$\mathbf{1} \cdot \pi_t^* = \mathbf{1} \cdot y\sigma_t^+\lambda_t$$

- Amount invested in the riskless asset

$$\pi_t^{0,*} = X_t^* - \mathbf{1} \cdot y\sigma_t^+\lambda_t$$

Such an allocation is rather **conservative** and is often viewed as an argument **against** the classical exponential criteria

Case 2: No benchmark and risk neutralization $\delta = 0$ and $\lambda + \phi = 0$

Then, $\mathcal{E}_t = 1$, $Y_t = y > 0$ and $Z_t = e^{-\int_0^t \frac{1}{2}|\lambda_s|^2 ds - \int_0^t \lambda_s \cdot dW_s}$

- Forward exponential performance process

$$U_t(x) = -\exp\left(-\frac{x}{y} - \frac{1}{2}\int_0^t |\lambda_s|^2 \, ds - \int_0^t \lambda_s \cdot dW_s\right)$$

- Optimal discounted wealth

$$X_t^* = x$$

Case 2: No benchmark and risk neutralization $\delta = 0$ and $\lambda + \phi = 0$ (continued)

- Optimal allocations

$$\pi_t^* = 0 \quad \text{and} \quad \pi_t^{0,*} = X_t^* = x$$

- Optimal exponential performance

$$U_t\left(X_t^*\right) = U_t\left(x\right)$$

$$= -\exp\left(-\frac{x}{y} - \frac{1}{2}\int_0^t \left|\lambda_s\right|^2 ds - \int_0^t \lambda_s \cdot dW_s\right)$$

It is important to notice that, for all trading times, the optimal allocation consists of putting **zero** into the risky assets and, therefore, investing the entire wealth into the riskless asset. Such a solution seems to capture quite accurately the strategy of a **derivatives trader** for whom the underlying **objective is to hedge** as opposed to the asset manager whose objective is to invest

Case 3: **Following the benchmark** $\delta = \lambda + \phi$ with $\lambda + \phi \neq 0$

Then $\delta = \sigma\sigma^{+}(\lambda + \phi)$ and $Z_t = \exp\left(-\int_0^t \frac{1}{2}|\phi_s|^2 ds + \int_0^t \phi_s \cdot dW_s\right)$

- Forward exponential performance process

$$U_t(x) = -\exp\left(-\frac{x}{Y_t} - \int_0^t \frac{1}{2}|\phi_s|^2 ds + \int_0^t \phi_s \cdot dW_s\right)$$

- Optimal wealth

$$X_t^* = x\mathcal{E}_t$$

- Returns of wealth and of benchmark

$$\frac{dX_t^*}{X_t^*} = \frac{dY_t}{Y_t}$$

Case 3: Following the benchmark $\delta = \lambda + \phi$ with $\lambda + \phi \neq 0$ (continued)

- Optimal allocation

$$\pi_t^* = \phi_s X_t^* \sigma_t^+ \delta_t$$

- Optimal exponential performance

$$U_t\left(X_t^*\right) = -\exp\left(-\frac{x}{Y_t} - \int_0^t \frac{1}{2}\left|\phi_s\right|^2 ds + \int_0^t \phi_s \cdot dW_s\right)$$

Observe that, contrary to what we have observed in traditional (backward) exponential utility problems, the optimal portfolio is a **linear functional of the wealth** and not independent of it

Case 4: **Generating arbitrary portfolio allocations**

- Assume that $\mathbf{1} \cdot \sigma_t^+ (\lambda_t + \phi_t) = 1$. Then

$$\mathbf{1} \cdot \pi_t^* = X_t^* \qquad \text{and} \qquad \pi_t^{0,*} = 0$$

Hence, the optimal allocation π^* puts **zero** amount in the riskless asset and invests **all** wealth in the risky assets, according to the weights specified by the vector $\sigma^+ (\lambda + \phi)$

Case 4: Generating arbitrary portfolio allocations (continued)

- Note, also, that for an arbitrary vector ν_t with $\mathbf{1}\cdot\sigma_t^{+}\nu_t \neq 0$, the vector

$$\phi_t = \frac{1-\mathbf{1}\cdot\sigma_t^{+}\lambda_t}{\mathbf{1}\cdot\sigma_t^{+}\nu_t}\nu_t$$

satisfies the above constraint since $\mathbf{1}\cdot\sigma_t^{+}\left(\lambda_t + \frac{1-\mathbf{1}\cdot\sigma_t^{+}\lambda_t}{\mathbf{1}\cdot\sigma_t^{+}\nu_t}\nu_t\right) = 1$

Can we generate optimal portfolios that allocate **arbitrary**, but constant, fractions of wealth to the different accounts?

The answer is affirmative. Indeed, for $p \in \mathcal{R}$, set,

$$\mathbf{1}\cdot\sigma_t^{+}\left(\lambda_t + \phi_t\right) = p$$

Then, the total investment in the risky assets and the allocation in the riskless bond are

$$\mathbf{1}\cdot\pi_t^{*} = pX_t^{*} \qquad \text{and} \qquad \pi_t^{0,*} = (1-p)\,X_t^{*}$$