# The Black-Scholes implied volatility at extreme strikes

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## Part I: The Tail-Wing Formula

Outline: - all the background needed on one page

- statement of result

- sketch of proof

- examples with known tail asymptotics

- see *www.arxiv.org/abs/math.PR/0603146*

- **Black-Scholes** normalized call price given by

$$c_{BS}(k,\sigma) = N(d_1) - e^k N(d_2)$$

where $k$ is log-strike and $d_{1,2}(k) = -k/\sigma \pm \sigma/2$.

- **Notation:** $F$ denotes the distribution function of risk-neutral returns, $\bar{F} \equiv 1 - F$ and (if $\exists$) $F' \equiv f$

- **Implied volatility** defined by

$$c_{BS}(k, V(k)) = \int_k^\infty \left(e^x - e^k\right) dF(x) \equiv c(k)$$

- **Definition:** $g$ regularly varying, index $\alpha$, $g \in R_\alpha$ iff

$$g(xt)/g(t) \to x^\alpha \qquad \text{as } t \to \infty.$$

- **Examples:** $t^2/2 \in R_2$, $t \log t \in R_1$, ...

- **Notation:** $g \sim h$ iff $g(t)/h(t) \to 1$ as $t \to \infty$.

**Tail-wing formula [Benaim, F]:** Assume $\alpha > 0$ and $\exists \epsilon > 0 : \mathbb{E}[e^{(1+\epsilon)X}] < \infty$ and define

$$\psi(x) \equiv 2 - 4\left(\sqrt{x^2 + x} - x\right).$$

Then

$$\text{(i)} \implies \text{(ii)} \implies \text{(iii)} \implies \text{(iv)},$$

where, always as $k \to \infty$,

$$-\log f(k) \in R_\alpha; \tag{i}$$

$$-\log \bar{F}(k) \in R_\alpha; \tag{ii}$$

$$-\log c(k) \in R_\alpha; \tag{iii}$$

and

$$V(k)^2/k \sim \psi[-\log c(k)/k]. \tag{iv}$$

If (ii) holds then $-\log c(k) \sim -k - \log \bar{F}$ and

$$V(k)^2/k \sim \psi\left[-1 - \log \bar{F}(k)/k\right], \tag{iv'}$$

if (i) holds, then $-\log f \sim -\log \bar{F}$ and

$$V(k)^2/k \sim \psi[-1 - \log f(k)/k]. \tag{iv''}$$

- There is a similar result for the left tail resp. wing.

- Note $\psi : [0,\infty] \searrow [0,2]$ and $\psi[x] \underset{x\to\infty}{\sim} 1/(2x)$.

- If $-1 - \log f(k)/k \to p^* \in (0,\infty)$ then

$$V(k)^2 \sim \psi(p^*) \times k \quad \text{(asymptotically linear)}$$

- If $-\log f(k)/k \to \infty$ then

$$V(k)^2 \sim \frac{1}{-2\log f(k)/k} \times k \text{ (asymptotically \textit{sub}linear)}$$

- **Sanity check:** Black-Scholes returns are Gauss with variance $\sigma^2$. Hence

$$-\log f_{BS}(k) \sim k^2/\left(2\sigma^2\right) \in R_2$$

From the tail-wing formula,

$$V(k)^2 \sim \frac{1}{-2\log f_{BS}(k)/k} \times k \sim \sigma^2$$

in trivial agreement with the flat smile $V \equiv \sigma$.

**Sketch of proof:** A motivating example,

$$-\log \int_x^\infty e^{-t^2/2} dt \sim x^2/2 \text{ as } x \to \infty$$

**Bingham's Lemma:** $g \in R_\alpha$ with $\alpha > 0$. Then

$$-\log \int_x^\infty e^{-g(t)} dt \sim g(x) \text{ as } x \to \infty.$$

**Claim 1:** $-\log \bar{F} \sim -\log f$.
*Apply Bingham's lemma to* $g = -\log f$.

**Claim 2:** $-\log c(k) \sim -k - \log \bar{F}(k)$.
*Apply Bingham's lemma after Integration by Parts*

$$c(k) = -\int_k^\infty \left(e^x - e^k\right) d\bar{F}(x) = \int_k^\infty e^x \bar{F}(x)\, dx.$$

**Claim 3:** $V(k)^2/k \sim \psi(-\log c(k)/k)$.
*Show that* $\log c(k) = -d_1^2/2 + O(\log k)$ *so that*

$$\frac{\log c(k)}{k} = -\frac{k}{2V(k)^2} + \frac{1}{2} - \frac{V(k)^2}{8k} + O\left(\frac{\log k}{k}\right)$$

*and solve for* $V(k)$.

**That's it!**

## Examples:

- **NIG Model:** $X = X_T \sim NIG(\alpha, \beta, \mu T, \delta T)$.
  We know that

  $$f(k) \sim C\,|k|^{-3/2}\, e^{-\sqrt{\beta^2+\gamma^2}|k|+\beta k} \text{ as } k \to \pm\infty$$

  Therefore $-\log f \in R_1$ and

  $$\log f(k)/k \to \left(-\sqrt{\beta^2+\gamma^2}+\beta\right) \text{ as } k \to +\infty.$$

  and the tail-wing formula gives

  $$\begin{aligned} \frac{\sigma_{BS}^2(k,T)\,T}{k} &\sim \psi\left(-1 - \log f(k)/k\right) \\ &\sim \psi\left(-1+\sqrt{\beta^2+\gamma^2}-\beta\right). \end{aligned}$$

- **FMLS Model:** $X = X_T \sim L_\alpha\left(\mu T, \sigma T^{1/\alpha}, -1\right)$ with $\alpha \in (1,2]$. Asymptotics of $\bar{F}$ known and imply

  $$-\log \bar{F}(k) \sim k^{\frac{\alpha}{\alpha-1}} \times \left[T\alpha\sigma^\alpha \left|\sec(\pi\alpha/2)\right|\right]^{-1/(\alpha-1)}.$$

  Note $-\log \bar{F} \in R_{\alpha/(\alpha-1)}$. From the tail-wing formula

  $$\sigma_{BS}^2(k,T)\,T \sim k^{1-\frac{1}{\alpha-1}} \times \frac{1}{2}\left[T\alpha\sigma^\alpha \left|\sec(\pi\alpha/2)\right|\right]^{1/(\alpha-1)},$$

  consistent with Black-Scholes as $\alpha \uparrow 2$.

- **Merton:** $X_\cdot$ is Lévy with triplet $(\mu, \sigma^2, K)$ where $K$ is $\lambda$ (=intensity of jump) times a Gaussian measure with mean $\alpha$ and standard deviation $\delta > 0$ describing the distribution of jumps. $\bar{F}(k)$ equals

  $$\mathbb{P}[X > k] \leq \inf_z e^{-zk}\mathbb{E}[\exp(zX)] = e^{K(z^*)-z^*k}$$

  where $K(z) = \log \mathbb{E}[\exp(zX)]$ and $z^* = z^*(k)$ is determined from $K'(z^*) = k$. Here

  $$K(z) = T\left\{z\mu + \frac{1}{2}z^2\sigma^2 + \lambda\left(e^{z\alpha+z^2\delta^2/2} - 1\right)\right\}$$

  from which $z^* = z^*(k) \sim \sqrt{2\log k}/\delta$ and

  $$\log \bar{F}(k) \leq K(z^*) - z^*k \sim -z^*k \sim -k\sqrt{2\log k}/\delta$$

  Nice Lévy tail estimates (Albin-Bengtsson, 2005) confirm

  $$\log \bar{F}(k) \sim -\frac{k}{\delta}\sqrt{2\log k}.$$

  From the tail-wing formula,

  $$\sigma_{BS}^2(k, T)\,T \sim \delta \times \frac{k}{2\sqrt{2\log k}}.$$

## Part II: Models with Known Moment Generating Functions

Outline: - link to Roger Lee's moment formula

- Tauberian theory

- Several Criteria

- Time Changed Lévy models

- numerical examples

- see *www.arxiv.org/abs/math.PR/0608619*

- What if only a moment generating function $M$ is known? **Roger Lee's moment formula** states that

$$\lim\sup_{k\to\infty} \frac{\sigma_{BS}^2(k,T)\,T}{k} = \psi(-1+r_T^*)$$

with critical exponent

$$r^* \equiv r_T^* \equiv \sup\{r \geq 0 : M(r) \equiv \mathbb{E}\exp(rX_T) < \infty\},$$

usually seen directly from explicitly known $M$.

- If $r^* = \infty$ the moment formula only says

$$\sigma_{BS}^2(k,T) = o(k).$$

In contrast, the tail-wing formula contains the full asymptotics (cf examples in Part I)

- Consider $r^* < \infty$. Numerical evidence that in all practical cases " $\lim\sup = \lim$ ", that is

$$\lim_{k\to\infty} \frac{\sigma_{BS}^2(k,T)\,T}{k} = \psi(-1+r_T^*) \qquad (1)$$

Can one prove (1) knowning the mgf $M$?

- Yes! By the Tail-Wing-Formula suffices to show

$$\log \bar{F}(k) \sim -r^* k \qquad (2)$$

and we have sufficient criteria for (2) that (seem to) cover all examples. (Remark: $\log \bar{F}(k) \lesssim -r^* k$ is easy.)

- **Criterion I:** $M$ or one of its derivatives (i.e. $M', M'', \ldots$) blows up in a regularly varying way at $r^*$.
  **Criterion II:** $\log M$ blows up in a regularly varying way at $r^*$.
  **Idea of proof**: Esscher-type change of measure followed by an application of Karamata's resp. Kohlbecker's Tauberian Theorem.
  ($\rightsquigarrow$ fine monograph by Bingham, Goldie, Teugels.)

## More Lévy Examples:

- **Variance Gamma:** $VG\,(m, g, CT)|_{T=1}$ has mgf

$$M(s) = \left(\frac{gm}{(m-s)(s+g)}\right)^C$$

See that $r^* = m$ and the $M$ satisfies criterion I:

$$M(r^* - s) \sim \left(\frac{gm}{m+g}\right)^C s^{-C} \text{ as } s \to 0+\,.$$

- **NIG Model (again!):** $NIG\,(\alpha, \beta, \mu T, \delta T)|_{T=1}$ has mgf

$$M\,(s) = \exp\left\{\delta\left\{\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + s)^2}\right\} + \mu s\right\}.$$

See that $r^* = \alpha - \beta$ and $M'$ satisfies criterion I:

$$M'(r^* - s) \sim 2\delta\alpha\sqrt{2\alpha}s^{-1/2}M(r^*) \text{ as } s \to 0+\,.$$

- **Kou's Double Exponential model** has mgf

$$\log M(s) = \frac{1}{2}\sigma^2 s^2 + \mu s + \lambda \left( \frac{p\eta_1}{\eta_1 - s} + \frac{q\eta_2}{\eta_2 + s} - 1 \right).$$

See that $r^* = \eta_1$ and $M$ satisfies criterion II:

$$\log M(\eta_1 - s) \sim \lambda p \eta_1 s^{-1} \text{ as } s \to 0+ .$$

- ... and in all Lévy examples $r^* \in (1, \infty)$ does not depend on $T$,

$$\forall T > 0 : \lim_{k \to \infty} \frac{\sigma_{BS}^2 (k, T) T}{k} = \psi(-1 + r^*).$$

## Time-changed Lévy Process:

- Lévy process $(L_t)$ $\leftrightsquigarrow$ cumulant generating fct of $L_1$ :

$$K_L(v) \equiv \log M_L(v) \equiv \log E\left[\exp\left(vL_1\right)\right]$$

Independent random clock $\tau = \tau(\omega, T) \geq 0$ with cgf $K_\tau = \log M_\tau$.
$\Longrightarrow$ Mgf of $L \circ \tau$ is given by

$$M(v) = M(v;T) = \exp\left[K_\tau(K_L(v)\right].$$

- **Theorem:** If both $M_L$ and $M_\tau$ satisfy one of the criteria, then $M$ does.
**Corollary:** $\log \bar{F}(k) \sim -r^* k$
**Corollary:** $\lim_{k\to\infty} \frac{\sigma^2_{BS}(k,T)T}{k} = \psi\left(-1 + r^*\right)$

- In practice, $K_{\mathcal{T}} = K_{\mathcal{T}}(\cdot, T), K_L$ known $\Longrightarrow$ $r^* = r^*_T$ easy to determine and have full analytic understanding of term structure of smile at extreme strikes,

$$T \mapsto \psi\left(-1 + r^*_T\right).$$

Example: **Variance Gamma with OU time change** with Schoutens et al. parameters. Plot $\sigma^2_{BS}(k, T)\, T$ for

$$T = 0.4,\ 0.9,\ 1.3.$$

The respective smile slopes $\psi\left(-1 + r^*_T\right)$ are

$$0.047,\ 0.053,\ 0.054 \in [0, 2].$$

## Part III: SABR, CEV: heat kernels, LDP

Outline: - the SABR model

- another (exponential) Tauberian theorem

- smile asymptotics

- geometry of SABR

- smile asymptotics revisited

- *preprint available on request*

- SABR model given by

$$\begin{aligned} dS &= \sigma S^{\beta} dW \\ d\sigma &= \eta \sigma dZ \\ \text{with } d\langle W, Z \rangle &= \rho dt, \quad \beta < 1, \eta > 0 \end{aligned}$$

- No explicit SABR density or mgf known.

- But: accurate asymptotic solution for implied vol in the at-the-money region (*Hagan's formula*).

- Wings more problematic ...

- Assuming $\rho = 0$ we can estimte $\mathbb{E}\left[S_T^r\right]$ as $r \to \infty$ well enough, using standard methods, to deduce the asymptotic behaviour of the mgf of $\log S_T$

$$\log \mathbb{E}\left[S_T^r\right] = \log \mathbb{E}\left[\exp\left(r \log S_T\right)\right] \sim \underbrace{\frac{\eta^2 T}{2\left(1-\beta\right)^2}}_{=:C} r^2$$

- *Kasahara's exponential Tauberian theorem* relates log-asymptotics of the mgf to log-asymptotics of the tail. Apply to $\log S_T$ :

$$-\log \bar{F}(k) \sim \frac{1}{4C}k^2 = \frac{(1-\beta)^2}{2\eta^2 T}k^2$$

and from the tail-wing formula

$$\begin{aligned}\frac{\sigma^2(k,T)\,T}{k} &\sim \psi\left[-1 - \log \bar{F}(k)/k\right] \\ &\sim \psi\left[\frac{(1-\beta)^2}{2\eta^2 T}k\right] \\ &\sim \left(2\frac{(1-\beta)^2}{2\eta^2 T}k\right)^{-1}\end{aligned}$$

so that

$$\sigma(k,T) \sim \frac{\eta}{(1-\beta)} \text{ as } k \to \infty.$$

- Consistent with working paper by V. Piterbarg (2006)

- The SABR Heat-kernel $p_t(S_0, \sigma_0; S, \sigma)$ can be obtained by solving forward PDE. Then

$$\mathbb{P}[S_t \in dS] = \int p_t(S_0, \sigma_0; S, \sigma)\, d\sigma.$$

- Classic result for diffusions (Varadhan): $p_t$ intimately related to a certain Riemannian metric (generator $\sim \frac{1}{2} a^{ij} \partial_{ij} \implies$ metric tensor $g = a^{-1}$.)

- $\exists$ conditions: $p_t$ is approximately Gaussian w.r.t. Riemannian distance $d_g$.

- Hagan, Lesniewski, Woodward (2005) use asymptotic methods to derive that the pdf of $S_t$ is approximately Gaussian with respect to distance

$$\begin{aligned} d(S_0, S) &= \frac{1}{\eta} \log \frac{\sqrt{\zeta^2 - 2\rho\zeta + 1} + \zeta - \rho}{1 - \rho}, \\ \zeta &= \frac{\eta}{\sigma_0} \int_{S_0}^{S} \frac{1}{u^\beta} du \sim \frac{\eta}{\sigma_0} \frac{S^{1-\beta}}{1 - \beta} \end{aligned}$$

To compare with our result above, let $\rho = 0$. Then, as $S \to \infty$

$$d(S_0, S) \sim \frac{1}{\eta} \log \zeta \sim \frac{(1-\beta)}{\eta} \log S$$

and

$$-\log \mathbb{P}[S_T \in dS] \approx \frac{1}{2T} d(S_0, S)^2 \sim \frac{(1-\beta)^2}{2\eta^2 T} (\log S)^2 .$$

Let $f$ denote the pdf of $\log S_T$. Then

$$-\log f(k) \sim \frac{(1-\beta)^2}{2\eta^2 T} k^2$$

This is consistent with $\log \bar{F}$-estimate obtained earlier using Kasahara's Tauberian theorem. The tail-wing formula gives the same asymptotic implied vol

$$\sigma(k, T) \sim \frac{\eta}{(1-\beta)} \text{ as } k \to \infty.$$

- CEV model = SABR with zero vvol, i.e.

$$dS = \sigma S^{\beta} dW, \;\; \beta < 1$$

- A (rare!) example where scaling links tail & short time behaviour

$$dS^{(K)} \equiv d\left(\frac{S}{K}\right) = \sigma\left(\frac{S}{K}\right)^{\beta}\frac{1}{K^{1-\beta}}dW \equiv \sigma S^{(K)}\varepsilon dW$$

and $\varepsilon = K^{\beta-1} \to 0$ as $K \to +\infty$. From

$$\varepsilon^2 \log \mathbb{P}\left[S_T^{(K)} > 1\right] \sim -\frac{1}{2}d^2(0,1)/T$$

i.e. where $d(0,1) = \int_0^1 \frac{1}{\sigma x^\beta}dx = 1/(1-\beta)\sigma$ so that

$$\log \mathbb{P}[S_T > K] \sim -\frac{1}{2}\frac{K^{2(1-\beta)}}{(1-\beta)^2\sigma^2 T}.$$

- Sanity check: $\beta = 0$ Bachelier model & see Gauss tail. Result confirmed via explicitly known density.

- Note that the right tail of $\log S_T$ is not regularly varying. Nonetheless, one can check that the conclusion of the tail-wing formula remains valid.

- Moral: tail-wing formula holds for all reasonable tails