

Optimal hedging under partial information

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Model

- $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}, P)$, and $\mathcal{F}_t = \sigma((B_s, B_s^\perp); 0 \leq s \leq t)$
- $S := (S_t)_{0 \leq t \leq T}$ traded

$$dS_t = rS_t dt + \sigma S_t(\lambda dt + dB_t) =: rS_t dt + \sigma S_t d\xi_t$$

- $Y := (Y_t)_{0 \leq t \leq T}$ non-traded

$$dY_t = rY_t dt + \beta Y_t(\theta dt + dW_t) =: rY_t dt + \beta Y_t d\zeta_t$$

- Correlation ρ

$$d[B, W]_t = \rho dt, \quad W = \rho B + \sqrt{1 - \rho^2} B^\perp, \quad \rho \in [-1, 1]$$

- European claim pays $h(Y_T)$

Perfect correlation case

- For $\rho = 1$, perfect hedging possible
- No arbitrage requires $\theta = \lambda$
- Position in n claims hedged by $\Delta_t^{(c)}$ units of S at $t \in [0, T]$

$$\Delta_t^{(c)} = -n \frac{\beta}{\sigma} \frac{Y_t}{S_t} \frac{\partial v}{\partial y}(t, Y_t)$$
$$v(t, y) = \text{BS}(t, y; \beta)$$

- Perfect hedge does not require knowledge of λ, θ

Incomplete market case ($\rho \neq 1$)

- Utility-based hedge, for $U(x) = -e^{-\alpha x}$, computable in analytic form
- Position in n claims hedged by Δ_t units of S at $t \in [0, T]$

$$\Delta_t = -n\rho \frac{\beta}{\sigma} \frac{Y_t}{S_t} \frac{\partial p}{\partial y}(t, Y_t)$$

$$p(t, y) = -\frac{e^{-r(T-t)}}{\alpha(1-\rho^2)n} \log E^{Q^M} [\exp(-\alpha(1-\rho^2)nh(Y_T)) | Y_t = y]$$

- Under Q^M

$$dY_t = rY_t dt + \beta Y_t [(\theta - \rho\lambda)dt + dW_t^M]$$

- Exponential hedge requires knowledge of λ, θ
- Superior to BS hedge if drifts known (Monoyios, 2004)

Drift estimation impossible

- Observe S over $[0, t]$ and estimate λ

$$\bar{\lambda}(t) = \frac{1}{t} \int_0^t \frac{d(e^{-rs} S_s)}{\sigma S_s} = \lambda + \frac{B_t}{t} \sim N(\lambda, 1/t)$$

- $(\bar{\lambda}(t) - \lambda)/(1/\sqrt{t}) \sim N(0, 1)$, and 95% confidence interval for λ is

$$[\bar{\lambda}_{\min}(t), \bar{\lambda}_{\max}(t)] = \left[\bar{\lambda}(t) - \frac{1.96}{\sqrt{t}}, \bar{\lambda}(t) + \frac{1.96}{\sqrt{t}} \right]$$

- Suppose true $\lambda = 1$. To be 95% sure of λ to within 5% of true value ($|\bar{\lambda}(t) - \lambda| \leq 0.05$) requires

$$\bar{\lambda}_{\max}(t) - \bar{\lambda}_{\min}(t) = 0.1 \Rightarrow t \approx 1537 \text{ years}$$

- Drift mis-estimation ruins indifference hedging (Monoyios, 2007)
- Can Bayesian learning help?

Partial information

- Require strategies to be adapted to observation filtration generated by asset returns
- Incorporate learning via filtering
- Use prior distribution for (λ, θ) , now considered as random variables
- Suppose (λ, θ) is bivariate normal

$$\lambda \sim N(\lambda_0, v_0), \quad \theta \sim N(\theta_0, \gamma_0), \quad \text{cov}(\lambda, \theta) = \kappa_0 = \rho\sqrt{v_0\gamma_0}$$

- Infer prior by observing S and Y over $[-\tau, 0]$
- Filter (update) estimates of λ, θ from subsequent observations of

$$\xi_t := \frac{1}{\sigma} \int_0^t \frac{d(e^{-rs} S_s)}{S_s} = \lambda t + B_t, \quad \zeta_t := \frac{1}{\beta} \int_0^t \frac{d(e^{-rs} Y_s)}{Y_s} = \theta t + W_t$$

over hedging interval $[0, T]$

Observation and signal

- Observation filtration

$$\hat{\mathbb{F}} := (\hat{\mathcal{F}}_t)_{0 \leq t \leq T}, \quad \hat{\mathcal{F}}_t = \sigma(\xi_s, \zeta_s; 0 \leq s \leq t)$$

- Observation process O , (unobservable) signal process U

$$O := \begin{pmatrix} \xi_t \\ \zeta_t \end{pmatrix}_{0 \leq t \leq T}, \quad U = \begin{pmatrix} \lambda \\ \theta \end{pmatrix}$$

- Observation and signal SDEs

$$dO_t = U dt + D dM_t, \quad dU = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1 - \rho^2} \end{pmatrix}, \quad M_t = \begin{pmatrix} dB_t \\ dB_t^\perp \end{pmatrix}$$

Kalman-Bucy equations

- Optimal filter $\hat{U}_t := E[U|\hat{\mathcal{F}}_t], 0 \leq t \leq T$

$$\hat{U}_t := \begin{pmatrix} E[\lambda|\hat{\mathcal{F}}_t] \\ E[\theta|\hat{\mathcal{F}}_t] \end{pmatrix} =: \begin{pmatrix} \hat{\lambda}_t \\ \hat{\theta}_t \end{pmatrix}, \quad \begin{pmatrix} \hat{\lambda}_0 \\ \hat{\theta}_0 \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ \theta_0 \end{pmatrix}$$

satisfies Kalman-Bucy filtering equation

$$d\hat{U}_t = V_t (DD^T)^{-1} (dO_t - \hat{U}_t dt) =: V_t dN_t$$

- $(N_t)_{0 \leq t \leq T}$ is innovations process

$$\begin{aligned} N_t &:= O_t - \int_0^t \hat{U}_s ds \\ &= \begin{pmatrix} \xi_t - \int_0^t \hat{\lambda}_s ds \\ \zeta_t - \int_0^t \hat{\theta}_s ds \end{pmatrix} =: \begin{pmatrix} \hat{B}_t \\ \hat{W}_t \end{pmatrix} \end{aligned}$$

and \hat{B}, \hat{W} are $\hat{\mathbb{F}}$ -BMs with correlation ρ

Riccati and Lyapunov equations

- Conditional variance-covariance matrix

$$V_t := E \left[(U - \hat{U}_t)(U - \hat{U}_t)^T \middle| \hat{\mathcal{F}}_t \right]$$

satisfies deterministic matrix Riccati equation

$$\frac{dV_t}{dt} = -V_t (DD^T)^{-1} V_t$$

- Equivalently, $R_t := V_t^{-1}$ satisfies Lyapunov equation

$$\frac{dR_t}{dt} = (DD^T)^{-1}$$

Expanded covariance matrix

- Written in full, conditional covariance is

$$\begin{aligned} V_t &= \begin{pmatrix} E[(\lambda - \hat{\lambda}_t)^2 | \hat{\mathcal{F}}_t] & E[(\lambda - \hat{\lambda}_t)(\theta - \hat{\theta}_t) | \hat{\mathcal{F}}_t] \\ E[(\lambda - \hat{\lambda}_t)(\theta - \hat{\theta}_t) | \hat{\mathcal{F}}_t] & E[(\theta - \hat{\theta}_t)^2 | \hat{\mathcal{F}}_t] \end{pmatrix} \\ &=: \begin{pmatrix} v_t & \kappa_t \\ \kappa_t & \gamma_t \end{pmatrix} \end{aligned}$$

Expanded filtering equation

- Equation for optimal filter is

$$\begin{aligned} \begin{pmatrix} d\hat{\lambda}_t \\ d\hat{\theta}_t \end{pmatrix} &= \frac{1}{1-\rho^2} \begin{pmatrix} v_t - \rho\kappa_t & \kappa_t - \rho v_t \\ \kappa_t - \rho\gamma_t & \gamma_t - \rho\kappa_t \end{pmatrix} \begin{pmatrix} d\xi_t - \hat{\lambda}_t dt \\ d\zeta_t - \hat{\theta}_t dt \end{pmatrix} \\ &= \frac{1}{1-\rho^2} \begin{pmatrix} v_t - \rho\kappa_t & \kappa_t - \rho v_t \\ \kappa_t - \rho\gamma_t & \gamma_t - \rho\kappa_t \end{pmatrix} \begin{pmatrix} d\hat{B}_t \\ d\hat{W}_t \end{pmatrix} \end{aligned}$$

Solution of Lyapunov equation

- Lyapunov equation is

$$\frac{d}{dt} \begin{pmatrix} \frac{\gamma_t}{v_t \gamma_t - \kappa_t^2} & -\frac{\kappa_t}{v_t \gamma_t - \kappa_t^2} \\ -\frac{\kappa_t}{v_t \gamma_t - \kappa_t^2} & \frac{v_t}{v_t \gamma_t - \kappa_t^2} \end{pmatrix} = \frac{1}{1 - \rho^2} \begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}$$

- Obtain 3 equations for γ_t, κ_t, v_t

$$\begin{aligned} \frac{\gamma_t}{v_t \gamma_t - \kappa_t^2} - \frac{\gamma_0}{v_0 \gamma_0 - \kappa_0^2} &= \frac{t}{1 - \rho^2} \\ \frac{\kappa_t}{v_t \gamma_t - \kappa_t^2} - \frac{\kappa_0}{v_0 \gamma_0 - \kappa_0^2} &= \frac{\rho t}{1 - \rho^2} \\ \frac{v_t}{v_t \gamma_t - \kappa_t^2} - \frac{v_0}{v_0 \gamma_0 - \kappa_0^2} &= \frac{t}{1 - \rho^2} \end{aligned}$$

Simplification

- Assume $\gamma_0 = v_0$
- Corresponds to having past observations over same time interval $[-\tau, 0]$ for both S and Y
- Then $\kappa_0 = \rho v_0$, and we obtain

$$v_t = \frac{v_0}{1 + v_0 t}, \quad \gamma_t = v_t, \quad \kappa_t = \rho v_t$$

Solution of filtering problem

- Equation for optimal filter simplifies to

$$\begin{pmatrix} d\hat{\lambda}_t \\ d\hat{\theta}_t \end{pmatrix} = v_t \begin{pmatrix} d\xi_t - \hat{\lambda}_t dt \\ d\zeta_t - \hat{\theta}_t dt \end{pmatrix} = v_t \begin{pmatrix} d\hat{B}_t \\ d\hat{W}_t \end{pmatrix}$$

so

$$\begin{pmatrix} \hat{\lambda}_t \\ \hat{\theta}_t \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ \theta_0 \end{pmatrix} + \int_0^t v_s \begin{pmatrix} d\hat{B}_s \\ d\hat{W}_s \end{pmatrix}$$

■ Equivalently

$$\begin{pmatrix} \hat{\lambda}_t \\ \hat{\theta}_t \end{pmatrix} \equiv \begin{pmatrix} \hat{\lambda}(t, \xi_t) \\ \hat{\theta}(t, \zeta_t) \end{pmatrix} \equiv \begin{pmatrix} \hat{\lambda}(t, S_t) \\ \hat{\theta}(t, Y_t) \end{pmatrix}$$

given by

$$\hat{\lambda}_t = \frac{\lambda_0 + v_0 \xi_t}{1 + v_0 t}, \quad \hat{\theta}_t = \frac{\theta_0 + v_0 \zeta_t}{1 + v_0 t}$$

with

$$\xi_t = \frac{1}{\sigma} \log \left(\frac{e^{-rt} S_t}{S_0} \right) + \frac{1}{2} \sigma t, \quad \zeta_t = \frac{1}{\beta} \log \left(\frac{e^{-rt} Y_t}{Y_0} \right) + \frac{1}{2} \beta t$$

Effective full information model

- Asset price SDEs under $\hat{\mathbb{F}}$ become

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t d\xi_t &= rS_t dt + \sigma S_t (\hat{\lambda}_t dt + d\hat{B}_t) \\ dY_t &= rY_t dt + \beta Y_t d\zeta_t &= rY_t dt + \beta Y_t (\hat{\theta}_t dt + d\hat{W}_t) \end{aligned}$$

- Full information model with random drift parameters $(\hat{\lambda}_t, \hat{\theta}_t)$
- Same result obtained by treating filtering of λ and θ as two one-dimensional Kalman filters with same prior variance
- Correlated asset Y gives no additional information about drift of S (and vice versa) under continuous observation unless past observations of S, Y are over different time intervals

Portfolio and wealth processes

- Trading strategy $\pi := (\pi_t)_{0 \leq t \leq T}$, $\hat{\mathbb{F}}$ -adapted, $\int_0^T \pi_t^2 dt < \infty$ a.s.
- Wealth process $X^\pi \equiv X := (X_t)_{0 \leq t \leq T}$

$$dX_t = rX_t dt + \sigma \pi_t (\hat{\lambda}_t dt + d\hat{B}_t)$$

- $U(x) = -\exp(-\alpha x)$, $x \in \mathbb{R}$, $\alpha > 0$
- Primal value function $u \equiv u^{(n)}$

$$u(t, x, s, y) = E[U(X_T + nh(Y_T)) | X_t = x, S_t = s, Y_t = y]$$

- May be expressed as $u(t, x, y; \hat{\lambda}_t, \hat{\theta}_t)$
- Optimal strategy $\pi^* \equiv \pi^{*,n}$
- Optimal wealth process by $X^* \equiv X^{*,n}$
- Random endowment $nh(Y_T)$ is bounded below

Indifference price and hedge

- Indifference price per claim at $t \in [0, T]$, given $X_t = x, S_t = s, Y_t = y$, is $p \equiv p^{(n)}$ given by

$$u^{(n)}(t, x - np^{(n)}(t, x, s, y), s, y) = u^{(0)}(t, x, s)$$

- Optimal hedging strategy $\pi^H := (\pi_t^H)_{0 \leq t \leq T}$

$$\pi_t^{(H)} := \pi_t^{*,n} - \pi_t^{*,0}, \quad 0 \leq t \leq T$$

Problem without claim, $n = 0$

- Dual value function for $n = 0$, $v(t, \eta) \equiv v(t, \eta; \hat{\lambda}_t)$

$$v(t, \eta) = E \left[V \left(\eta \frac{\tilde{Z}_T}{Z_t} \right) \middle| \hat{\mathcal{F}}_t \right]$$

where

$$\tilde{Z}_t = e^{-rt} Z_t, \quad Z_t = \mathcal{E}(-\hat{\lambda} \cdot \hat{B})_t$$

This gives

$$v(t, \eta) = e^{-r(T-t)} \left[V(\eta) + \frac{\eta}{\alpha} \left(-r(T-t) + H^{(0)}(t, \hat{\lambda}_t) \right) \right]$$

where

$$H^{(0)}(t, \hat{\lambda}_t) = E \left[\frac{Z_T}{Z_t} \log \left(\frac{Z_T}{Z_t} \right) \middle| \hat{\mathcal{F}}_t \right]$$

is entropic term

Entropy computation

- Apply Itô to $Z_t =: g(t, \hat{\lambda}_t)$ and use

$$dZ_t = -\hat{\lambda}_t Z_t d\hat{B}_t, \quad d\hat{\lambda}_t = v_t d\hat{B}_t,$$

to obtain

$$\frac{Z_T}{Z_t} = \left(\frac{v_t}{v_T}\right)^{1/2} \exp\left[-\frac{1}{2}\left(\frac{\hat{\lambda}_T^2}{v_T} - \frac{\hat{\lambda}_t^2}{v_t}\right)\right]$$

- Conditional on $\hat{\mathcal{F}}_t$, $\hat{\lambda}_T \sim N(\hat{\lambda}_t, v_t v_T (T - t))$, so

$$H^{(0)}(t, \hat{\lambda}_t) = \frac{1}{2} \left[\log\left(\frac{v_t}{v_T}\right) - v_T (T - t) + \frac{\hat{\lambda}_t^2}{v_t} \right]$$

Primal value function for $n = 0$

- Recover $u(t, x) \equiv u(t, x; \hat{\lambda}_t)$ from bidual relation

$$u(t, x) = \inf_{\eta > 0} [v(t, \eta) + x\eta]$$

so for $n = 0$, $u \equiv u^{(0)}$ given by

$$u(t, x; \hat{\lambda}_t) = -\exp \left\{ -\alpha x e^{r(T-t)} - H^{(0)}(t, \hat{\lambda}_t) \right\}$$

- Recall $\hat{\lambda}_t \equiv \hat{\lambda}(t, S_t)$ so, conditional on $S_t = s$, $u(t, x; \hat{\lambda}_t) \equiv u(t, x, s)$

Optimal trading strategy ($n = 0$)

- Given $X_0 = x$, and writing $u(x) \equiv u(0, x)$

$$U'(X_T^*) = \eta e^{-rT} Z_T, \quad \eta = u'(x)$$

- Combine with Q -martingale property of $(e^{-rt} X_t^*)_{0 \leq t \leq T}$, and $dQ/dP = Z_T$,

$$\begin{aligned} X_t^* &= e^{-r(T-t)} E \left[\frac{Z_T}{Z_t} X_T^* \middle| \hat{\mathcal{F}}_t \right] \\ &= x e^{rt} + \frac{e^{-r(T-t)}}{\alpha} \left(H^{(0)}(0, \lambda_0) - H^{(0)}(t, \hat{\lambda}_t) - \log Z_t \right) \end{aligned}$$

Compute dX_t^* via Itô, recall $dX_t^* = rX_t^* dt + \sigma \pi_t^* (\hat{\lambda}_t dt + d\hat{B}_t)$:

$$\pi_t^* = e^{-r(T-t)} \frac{\hat{\lambda}_t}{\sigma \alpha} \frac{v_T}{v_t}$$

Problem with claim, $n \neq 0$

- Dual value function now $v(t, \eta, y) \equiv v(t, \eta, y; \hat{\lambda}_t, \hat{\theta}_t) \equiv v(t, \eta, s, y)$

$$v(t, \eta, y) = \inf_{Q \in \mathcal{M}} E \left[V \left(\eta \frac{\tilde{Z}_T^Q}{\tilde{Z}_t^Q} \right) + \eta \frac{\tilde{Z}_T^Q}{\tilde{Z}_t^Q} \middle| \hat{\mathcal{F}}_t \right]$$

where now $\tilde{Z}_t^Q = e^{-rt} Z_t^Q$ and

$$Z_t^Q = \frac{dQ}{dP} \bigg|_{\hat{\mathcal{F}}_t} = \mathcal{E}(-\hat{\lambda} \cdot \hat{B} - \psi \cdot \hat{B}^\perp)_t$$

and under $Q \in \mathcal{M}$, with $\epsilon := \sqrt{1 - \rho^2}$,

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t d\hat{B}_t^Q \\ dY_t &= rY_t dt + \beta Y_t [(\hat{\theta}_t - \rho \hat{\lambda}_t - \epsilon \psi_t) dt + d\hat{W}_t^Q] \\ d\hat{\lambda}_t &= v_t [-\hat{\lambda}_t dt + d\hat{B}_t^Q] \\ d\hat{\theta}_t &= v_t [-(\rho \hat{\lambda}_t + \epsilon \psi_t) dt + d\hat{W}_t^Q] \end{aligned}$$

Dual value function, $n \neq 0$

- This gives

$$v(t, \eta, y) = e^{-r(T-t)} \left[V(\eta) + \frac{\eta}{\alpha} \left(-r(T-t) + H^{(n)}(t, y; \hat{\lambda}_t, \hat{\theta}_t) \right) \right]$$

where

$$H^{(n)}(t, y; \hat{\lambda}_t, \hat{\theta}_t) = \inf_{Q \in \mathcal{M}} E^Q \left[\log \left(\frac{Z_T^Q}{Z_t^Q} \right) + \alpha n h(Y_T) \middle| \hat{\mathcal{F}}_t \right]$$

is entropic term

Entropic term

- $H^{(n)}$ decomposes according to

$$\begin{aligned} H^{(n)}(t, y; \hat{\lambda}_t, \hat{\theta}_t) &= E^Q \left[\frac{1}{2} \int_t^T \hat{\lambda}_u^2 du \middle| \hat{\mathcal{F}}_t \right] + C(t, y; \hat{\lambda}_t, \hat{\theta}_t) \\ &= H^{(0)}(t, \hat{\lambda}_t) + C(t, y; \hat{\lambda}_t, \hat{\theta}_t) \end{aligned}$$

where $C(t, y; \hat{\lambda}_t, \hat{\theta}_t) \equiv C(t, s, y)$ is value function of stochastic control problem

$$C(t, s, y) := \inf_{\psi} E^Q \left[\frac{1}{2} \int_t^T \psi_u^2 du + \alpha nh(Y_T) \middle| S_t = s, Y_t = y \right]$$

HJB equation and optimal control

- HJB equation for $C(t, s, y)$ is

$$C_t + \mathcal{A}_{S,Y}^M C + \min_{\psi} \left[\frac{1}{2} \psi^2 - \beta \epsilon \psi y C_y \right] = 0 \quad C(T, s, y) = \alpha n h(y)$$

where $\mathcal{A}_{S,Y}^M$ is generator of (S, Y) under minimal measure

$$\mathcal{A}_{S,Y}^M C = r s C_s + \frac{1}{2} s^2 C_{ss} + (r + \beta(\hat{\theta} - \rho \hat{\lambda})) y C_y + \frac{1}{2} \beta^2 y^2 C_{yy} + \rho \sigma \beta s y C_{sy}$$

- Optimal control is $\psi_t^* \equiv \psi^*(t, S_t, Y_t)$ where

$$\psi^*(t, s, y) = \epsilon \beta y C_y(t, s, y)$$

so HJB equation is semi-linear PDE

$$C_t + \mathcal{A}_{S,Y}^M C - \frac{1}{2} (1 - \rho^2) \beta^2 y^2 C_y^2 = 0$$

Primal value function and indifference price

- Primal value function recovered via bidual relation

$$u(t, x, s, y) = -\exp \left\{ -\alpha x e^{r(T-t)} - H^{(0)}(t, s) - C(t, s, y) \right\}$$

- Indifference price $p \equiv p^{(n)}$

$$p^{(n)}(t, s, y) = \frac{e^{-r(T-t)}}{\alpha n} C(t, s, y)$$

satisfies

$$p_t + \mathcal{A}_{S, Y}^M p - rp - \frac{1}{2} \alpha n (1 - \rho^2) e^{r(T-t)} \beta^2 y^2 p_y^2 = 0, \quad p(T, s, y) = h(y)$$

Optimal hedge

- HJB equation for primal value function gives π^* in terms of derivatives of $C(t, s, y)$
- Optimal hedge $(\Delta_t^H)_{0 \leq t \leq T}$ follows as

$$\Delta_t^H = -n \left(p_s^{(n)}(t, S_t, Y_t) + \rho \frac{\beta}{\sigma} \frac{Y_t}{S_t} p_y^{(n)}(t, S_t, Y_t) \right)$$

Residual risk

- Trade n claims at time 0 for $p^{(n)}(0, Y_0, S_0)$ per claim and hedge with strategy $(\Delta_t^H)_{0 \leq t \leq T}$
- Overall position worth $L = (L_t)_{0 \leq t \leq T}$ given by

$$L_t = X_t^H + np^{(n)}(t, S_t, Y_t), \quad L_0 = 0$$

- Itô and PDE satisfied by $p \equiv p^{(n)}$ gives

$$\begin{aligned} dL_t &= rL_t dt + n \left[\beta \left(\hat{\theta}_t - \rho \hat{\lambda}_t - (\theta - \rho \lambda) \right) Y_t p_y(t, S_t, Y_t) \right. \\ &\quad \left. + \frac{1}{2} \alpha n (1 - \rho^2) e^{r(T-t)} \beta^2 Y_t^2 (p_y)^2(t, S_t, Y_t) \right] dt \\ &\quad + n \sqrt{1 - \rho^2} \beta Y_t p_y(t, S_t, Y_t) dB_t^\perp \end{aligned}$$

Approximation for indifference price

- Indifference price given $S_t = s, Y_t = y$

$$p(t, s, y) = \frac{e^{-r(T-t)}}{\alpha n} \inf_{\psi} E^Q \left[\frac{1}{2} \int_t^T \psi_u^2 du + \alpha n h(Y_T) \middle| S_t = s, Y_t = y \right]$$

$$dS_t = rS_t dt + \sigma S_t d\hat{B}_t^Q$$

$$dY_t = rY_t dt + \beta Y_t [(\hat{\theta}_t - \rho \hat{\lambda}_t - \epsilon \psi_t) dt + d\hat{W}_t^Q]$$

$$d\hat{\lambda}_t = v_t [-\hat{\lambda}_t dt + d\hat{B}_t^Q]$$

$$d\hat{\theta}_t = v_t [-(\rho \hat{\lambda}_t + \epsilon \psi_t) dt + d\hat{W}_t^Q]$$

- For small ϵ approximate optimal control by $\psi = 0$, so trade claim at marginal price $\hat{p}(0, S_0, Y_0)$ where

$$\hat{p}(t, s, y) = e^{-r(T-t)} E^M [h(Y_T) | S_t = s, Y_t = y]$$

Distribution of Y_T under Q^M

- Marginal price \hat{p} computable in analytic form
- Under Q^M , conditional on $S_t = s, Y_t = y$, $\log Y_T$ is Gaussian

$$\begin{aligned}\log Y_T &\sim N(m(t, s, y)(T - t), \Sigma^2(t)(T - t)) \\ m(t, s, y) &= \log y + r + \beta(\hat{\theta}_t - \rho\hat{\lambda}_t - \frac{1}{2}\beta) \\ \Sigma^2 &= \beta^2(1 + (1 - \rho^2)v_t(T - t))\end{aligned}$$

- Allows fast simulation of L , which now satisfies

$$\begin{aligned}dL_t &= rL_t dt + n\beta \left(\hat{\theta}_t - \rho\hat{\lambda}_t - (\theta - \rho\lambda) \right) Y_t p_y(t, S_t, Y_t) dt \\ &+ n\sqrt{1 - \rho^2}\beta Y_t p_y(t, S_t, Y_t) dB_t^\perp\end{aligned}$$

How effective is optimal hedge with learning?

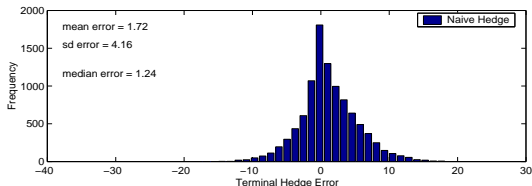
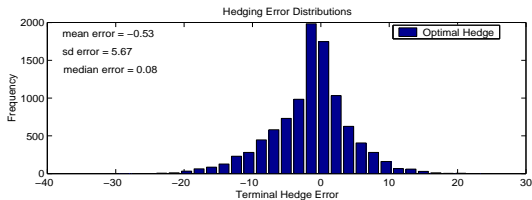
- Compare terminal residual risk with that from using by BS-style hedge, given by

$$\begin{aligned} dL_t^{\text{BS}} &= rL_t^{\text{BS}} dt + n\beta Y_t(\theta - \lambda)v_y(t, Y_t)dt \\ &+ n\beta Y_t v_y(t, Y_t)[(\rho - 1)dB_t + \sqrt{1 - \rho^2}dB_t^\perp] \end{aligned}$$

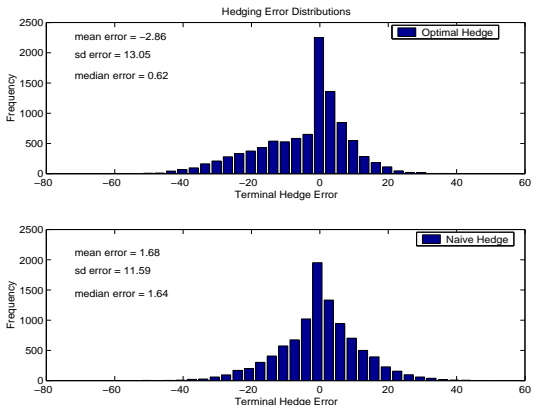
- For each simulation, obtain λ_0, θ_0 as point estimates of λ, θ over past data (use $[-\tau, 0]$, so $v_0 = 1/\tau$), then update using filtering results

Typical result over 10,000 paths, short put

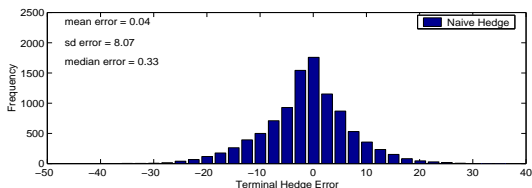
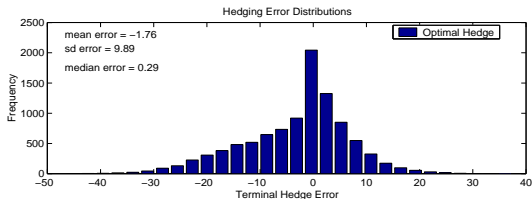
- $\lambda = 0.25, \theta = 0.4, \rho = 0.95, K = 100$
- $\langle S_0 \rangle = 82.9, \langle Y_0 \rangle = 87.2$
- $\langle \hat{\rho} \rangle = 16.0, \langle \text{BS} \rangle = 17.2$



- $\lambda = 0.25, \theta = 0.4, \rho = 0.6, K = 100$
- $\langle S_0 \rangle = 82.8, \langle Y_0 \rangle = 87.1$
- $\langle \hat{p} \rangle = 18.0, \langle \text{BS} \rangle = 17.2$



- $\lambda = 0.39, \theta = 0.4, \rho = 0.75, K = 100$
- $\langle S_0 \rangle = 84.3, \langle Y_0 \rangle = 85.1$
- $\langle \hat{p} \rangle = 16.9, \langle \text{BS} \rangle = 15.9$



Conclusions

- BS-style hedging robust
- Filtering and learning not sufficient to handle parameter uncertainty in utility-based hedge
- Exact numerical results seem unlikely to alter conclusion