A model for large investor, where she trades at utility indifference prices of market makers

Dmitry Kramkov (joint work with Peter Bank)

Carnegie Mellon University and Univeristy of Oxford

Further Developments in Quantitative Finance, ICMS, Edinburgh, July 9–13, 2007

Outline

Standard financial model (for “small” trader)

Desirable features of a model for “large” trader

Literature (very incomplete!)

Trading at utility indifference prices of the market

Asymptotic analysis: summary of results

Conclusion

Model for "small" trader

Input: price process $S = (S_t)$.

Key assumption: trader's actions do not affect S.

For *simple* strategy

$$\theta_t = \sum_{n=1}^{N} \vartheta_n 1_{(t_n, t_{n+1}]},$$

where θ_t is the number of stocks at time t and $\vartheta_n \in \mathbf{L}^0(\mathcal{F}_{t_n})$, the terminal wealth

$$X_T(\theta) = \sum_{n=1}^{N} \vartheta_n (S_{t_{n+1}} - S_{t_n})$$

Mathematical challenge: define terminal wealth for *general* θ.

Passage to continuous time trading

The passage from discrete to continuous-time trading is done in two steps:

1. Establish that S is a *semimartingale*
   1.1 $\Leftrightarrow \exists$ limit of discrete sums, when the sequence (θ^n) of simple integrand converges uniformly (Bechteler-Dellacherie)
   1.2 $\Leftarrow$ Absence of arbitrage for *simple strategies* (NFLBR) (Delbaen & Schachermayer (1994)).
2. If S is a semimartingale, then we can extend the map $\theta \to X_T(\theta)$ from simple to general (continuous) strategies θ arriving to *stochastic integrals*:

$$X_T(\theta) = \int_0^T \theta_t dS_t.$$

Basic results for the “small” trader model

Fundamental Theorems of Asset Pricing:

1. Absence of arbitrage for general admissible strategies (NFLVR) $\Leftrightarrow$ S is a local martingale under an equivalent probability measure (Delbaen & Schachermayer (1994)).
2. Completeness $\Leftrightarrow$ Uniqueness of a martingale measure for S (Harrison & Pliska (1983)).

Arbitrage-free pricing formula: in complete financial model the arbitrage-free price for a European option with maturity T and payoff f is given by

$$p = \mathbb{E}_{\mathbb{Q}}[f],$$

where $\mathbb{Q}$ is the unique martingale measure.

Desirable features of a model for "large" trader

Logical requirements:

1. Allow for general continuous-time trading strategies.
2. Give the "small" trader model in the limit:

$$X_T(\epsilon\theta) = \epsilon \int_0^T \theta_t dS_t^{(0)} + o(\epsilon), \quad \epsilon \to 0.$$

Practical goal: computation of *liquidity corrections* to the prices of derivatives:

$$p(\epsilon) = \epsilon \mathbb{E}_{\mathbb{Q}}[g] + \underbrace{\frac{1}{2}\epsilon^2 C(g)}_{\text{liquidity correction}} + o(\epsilon^2).$$

Here $p(\epsilon)$ is the price for ϵ contingent claims g. Of course, we expect to have

$$C(g) \leq 0 \text{ for all } g \text{ and } < 0 \text{ for some } g.$$

Literature (very incomplete!)

Model is an input: Jarrow (1992), (1994);
Frey and Stremme (1997);
Platen and Schweizer (1998);
Papanicolaou and Sircar (1998);
Cuoco and Cvitanic (1998);
Cvitanic and Ma (1996);
Schonbucher and Wilmott (2000);
Cetin, Jarrow and Protter (2002);
Bank and Baum (2003);
Cetin, Jarrow, Protter and Warachka (2006),
...

Model is an output (a result of equilibrium):
Kyle (1985), Back (1990), ...

Financial model

1. Uncertainty and the flow of information are modelled, as usual, by a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$.
2. Traded securities are European contingent claims with maturity T and payments $f = (f_i)$.
3. Prices are quoted by a finite number of market makers.
   3.1 Utility functions $U = (U_m(x))_{x \in \mathbf{R}, 1 \leq m \leq M}$ (defined on *real line*):

$$\frac{1}{c} < -\frac{U'_m(x)}{U''_m(x)} < c \text{ for some } c > 0.$$

   $\Rightarrow$ U_m has exp-like behavior. In particular, U_m is bounded above and we can assume that

$$U_m(\infty) = 0.$$

   3.2 Initial (random) endowments $E_0 = (E_0^m)_{1 \leq m \leq M}$ ($\mathcal{F}$-measurable random variables) form a *Pareto allocation*.

Pareto allocation

Definition

A vector of random variables $E = (E^m)_{1 \le m \le M}$ is called *a Pareto allocation* if there is no other allocation $F = (F^m)_{1 \le m \le M}$ of the same total endowment:

$$\sum_{m=1}^{M} F^m = \sum_{m=1}^{M} E^m,$$

which would leave all market makers not worse and at least one of them better off in the sense that

$$\mathbb{E}[U_m(F^m)] \ge \mathbb{E}[U_m(E^m)] \quad \text{for all } 1 \le m \le M,$$

and

$$\mathbb{E}[U_m(F^m)] > \mathbb{E}[U_m(E^m)] \text{ for some } 1 \le m \le M.$$

Pricing measure of Pareto allocation

First-order condition: We have equivalence between

1. $E = (E^m)_{1 \leq m \leq M}$ is a Pareto allocation.
2. The ratios of the marginal utilities are non-random:

$$\frac{U'_m(E^m)}{U'_n(E^n)} = \text{const}(m, n).$$

Pricing measure $\mathbb{Q}$ of the Pareto allocation E is defined by the marginal rate of substitution rule:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{U'_m(E^m)}{\mathbb{E}[U'_m(E^m)]}, \quad 1 \leq m \leq M.$$

(Marginal) price process of the traded contingent claims f corresponding to the Pareto allocation E is defined to be

$$S_t = \mathbb{E}_{\mathbb{Q}}[f | \mathcal{F}_t]$$

A trading of very small quantities at this price does not change the expected utilities of market makers.

Simple strategy

Strategy: a predictable process $\theta = (\theta^i_t)$, where θ^i_t is the number of the contingent claims f_i in the portfolio at time t.

We need to specify the terminal wealth $X_T(\theta)$, that is, to define the map

$$\theta \to X_T(\theta).$$

We start with a *simple* strategy

$$\theta_t = \sum_{n=1}^{N} \vartheta_n 1_{(t_n, t_{n+1}]},$$

Denote by $\eta = (\eta_t)$ the corresponding *cash balance process*:

$$\eta_t = \sum_{n=1}^{N} \xi_n 1_{[t_n, t_{n+1})}.$$

Here ϑ_n and ξ_n are $\mathcal{F}_{\tau_n}$-measurable r.v. denoting the cumulative number of shares and the amount of cash received by the agent up to and including time t_n.

Trading at time 0

1. The market makers start with the initial Pareto allocation $E_0 = (E_0^m)_{1 \leq m \leq M}$ of the total (random) endowment:

$$\Sigma_0 := \sum_{m=1}^{M} E_0^m.$$

2. After the trade in ϑ_0 shares at the cost ξ_0, the total endowment becomes

$$\Sigma_{t_1} = \Sigma_0 - \xi_0 - \vartheta_0 f.$$

3. Σ_{t_1} is redistributed as *Pareto allocation* $E_{t_1} = (E_{t_1}^m)_{1 \leq m \leq M}$.
4. **Key condition:** *the expected utilities of market makers do not change*, that is,

$$\mathbb{E}[U_m(E_{t_1}^m)] = \mathbb{E}[U_m(E_0^m)], \quad 1 \leq m \leq M.$$

Trading at time t_n

1. The market makers have $\mathcal{F}_{t_{n-1}}$-Pareto allocation $E_{t_n} = (E^m_{t_n})_{1\leq m\leq M}$ of the total (random) endowment:
$$\Sigma_{t_n} := \sum_{m=1}^{M} E^m_{t_n} = \Sigma_0 - \xi_{n-1} - \vartheta_{n-1} f.$$
2. After the trade in $\vartheta_n - \vartheta_{n-1}$ shares at the cost $\xi_n - \xi_{n-1}$, the total endowment becomes
$$\begin{aligned} \Sigma_{t_{n+1}} =& \Sigma_{t_n} - (\xi_n - \xi_{n-1}) - (\vartheta_n - \vartheta_{n-1}) f \\ =& \Sigma_0 - \xi_n - \vartheta_n f. \end{aligned}$$
3. $\Sigma_{t_{n+1}}$ is redistributed as $\mathcal{F}_{t_n}$-*Pareto allocation* $E_{t_{n+1}}$.
4. **Key condition:** *the conditional expected utilities of market makers do not change*, that is,
$$\mathbb{E}[U_m(E^m_{t_{k+1}})|\mathcal{F}_{t_k}] = \mathbb{E}[U_m(E^m_{t_k})|\mathcal{F}_{t_k}], \quad 1 \leq m \leq M.$$

Final step

The large trader arrives at maturity $t_N = T$ with

1. $\theta_T = \vartheta_N$ shares of contingent claims f.
2. cash amount $\eta_T = \xi_N$.

Hence, finally, her terminal wealth is given by

$$X_T(\theta) := \eta_T + \theta_T f = \Sigma_0 - \Sigma_T.$$

Lemma

For any simple strategy θ the cash balance process $\eta = \eta(\theta)$ and the terminal wealth $X_T(\theta)$ are defined uniquely.

Mathematical challenge: define the terminal wealth $X_T(\theta)$ for *general* strategy θ.

More on economic assumptions

This wealth evolution for large trader is essentially based on two economic assumptions:

Market efficiency After each trade the market makers form a *Pareto allocation*.
$\Leftrightarrow$ They can trade any contingent claim between each other (not only f)!

Information At any moment, the market makers do not anticipate any future trades on behalf of the large economic agent.
$\Leftrightarrow$ The agent can split any order in a sequence of very small trades at marginal prices.
$\Leftrightarrow$ The expected utilities of market makers do not change.

Remark

From the standpoint of investor this is the most “friendly” type of interaction with market makers.

Comparison with Arrow-Debreu equilibrium

Economic assumptions behind Arrow-Debreu equilibrium:

Market efficiency (Same as above)
After re-balance the market makers form a *Pareto allocation*.
$\Leftrightarrow$ They can trade anything between each other (not only f)!

Information The market makers have *perfect knowledge* of strategy θ.
$\Leftrightarrow$ The change in Pareto allocation occurs only at initial time.
$\Rightarrow$ The expected utilities of market makers *increase* as the result of trade.

Large trader model based on Arrow-Debreu equilibrium

Given a strategy θ the market makers immediately change the initial Pareto allocation E_0 to another Pareto allocation $\widetilde{E} = \widetilde{E}(\theta)$ with pricing measure $\widetilde{\mathbb{Q}}$, the price process

$$\widetilde{S}_t := \mathbb{E}_{\widetilde{\mathbb{Q}}}[f|\mathcal{F}_t]$$

and total endowment

$$\widetilde{\Sigma} := \sum_{m=1}^{M} \widetilde{E}^m$$

such that

$$\Sigma_0 - \widetilde{\Sigma} = \int_0^T \theta d\widetilde{S},$$

and the following “clearing” conditions hold true:

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}[E_0^m] = \mathbb{E}_{\widetilde{\mathbb{Q}}}[\widetilde{E}^m], \quad 1 \leq m \leq M.$$

Technical assumptions

Back to our model.

Mathematical challenge: define terminal wealth for *general* θ.

Assumption

The utility functions of market makers have bounded *prudence* coefficient:

$$\left| -\frac{U'''(x)}{U''(x)} \right| \leq K, \text{ for some constant } K > 0.$$

Assumption

The filtration is generated by a Brownian motion $W = (W^i)$ and the *Malliavin derivatives* of the payoffs $f = (f^k)$ are bounded:

$$|\mathbf{D}_t(f)| < K, \quad 0 \leq t \leq T, \text{ for some constant } K > 0.$$

Process of indirect utilities

For a simple strategy θ we denote by $u = (u_t^m(\theta))$ the process of expected (indirect) utilities for market makers:

$$u_t^m = \mathbb{E}[U_m(E_t^m)|\mathcal{F}_t], \quad 0 \leq t \leq T.$$

Clearly, the knowledge of u_T^m allows us to compute the terminal endowment E_T^m

$$u_T^m = U_m(E_T^m) \Rightarrow E_T^m = U_m^{-1}(u_T^m)$$

and, hence, the terminal wealth of large economic agent:

$$X_T(\theta) = \Sigma_0 - \Sigma_T = \Sigma_0 - \sum_{m=1}^{M} U_m^{-1}(u_T^m).$$

Crucial observation: for a simple strategy θ at any time t the knowledge of indirect utilities $u_t = (u_t^m)$ and the number of stocks θ_t allows us to identify uniquely the Pareto allocation $E_t(\theta)$.

Passage to continuous-time trading

The key intermediate result is the following

Theorem

Assume the technical conditions above. There is a continuously differentiable stochastic vector field $A = (A_t(u, q))$ *and a constant* $K > 0$ *such that*

$$|A_t^m| \leq K|u_m|(1 + |q|)$$
$$|\frac{\partial A_t^m}{\partial u_k}| \leq K\frac{u_m}{u_k}(1 + |q|)$$

and for any simple strategy θ *the indirect utilities of the market makers solve the following stochastic differential equation:*

$$du_t = A_t(u_t, \theta_t)dW_t, \quad u_0^m = \mathbb{E}[U_m(E_0^m)].$$

Passage to continuous-time trading

Theorem

Assume the technical conditions above. Let (θ^n) *be a sequence of simple processes and* θ *be a (general) stochastic process such that*

1. $|\theta^n| \leq \Psi$, *where* $\Psi = (\Psi_t)$ *be a stochastic process such that*

$$\int_0^T \Psi_u^2 du < \infty$$

2. (θ^n) *converges to* θ *in* $\mathbb{P}(d\omega) \times dt$*-probability.*

Then the terminal capitals $X_T(\theta^n)$ *converge in probability to*

$$X_T(\theta) = \Sigma_0 - \Sigma_T = \sum_{m=1}^{M} E_0^m - \sum_{m=1}^{M} U_m^{-1}(u_T^m)$$

where $u = u(\theta)$ *solves the following SDE:*

$$du_t = A_t(u_t, \theta_t) dW_t, \quad u_0^m = \mathbb{E}[U_m(E_0^m)].$$

Remark on admissibility

The previous theorem allows us to define terminal wealth for any process θ satisfying:

$$\int_0^T \theta_t^2 dt < \infty \quad (\mathbb{P} - a.s.).$$

Contrary to classical "small" agent model this set of strategies does not allow arbitrage. Indeed, for any such θ the process of indirect utilities $u = u(\theta)$ is a *local martingale bounded above (by* 0*)* $\Rightarrow$ a global *submartingale*. In particular,

$$\mathbb{E}[U_m(E_T^m(\theta))] \geq \mathbb{E}[U_m(E_0^m)], \quad 1 \leq m \leq M,$$

and, therefore, if the terminal wealth of the large agent

$$X_T(\theta) = \sum_{m=1}^M E_0^m - \sum_{m=1}^M E_T^m(\theta)$$

is nonnegative ($\mathbb{P}$-a.s.), then it is given by 0.

Asymptotic analysis: summary of results

- For a strategy θ we have the following expansion for terminal wealth:
$$X_T(\epsilon\theta) = \epsilon \int_0^T \theta_u dS_u^0 + \frac{1}{2}\epsilon^2 L_T(\theta),$$
where $L_T(\theta)$ can be computed by solving two auxiliary *linear* SDEs.
- We use above expansion to compute replication strategy and liquidity correction to the prices of derivatives in the next order (ϵ^2). (Good qualitative properties!)
- Key inputs: *risk-tolerance wealth processes* of market makers for initial Pareto equilibrium.

Conclusion

- We have developed a continuous-time model for large trader starting with economic primitives, namely, the preferences of market makers.
- In this model, the large investor trades "smartly", not revealing herself to market makers and, hence, not increasing their expected utilities.
- We show that the computation of terminal wealth $X_T(\theta)$ for a strategy θ comes through a solution of a non-linear SDE.
- The model allows us to compute rather explicitly liquidity corrections to the terminal capitals of trading strategies and to the prices of derivatives.
- The model has "good" qualitative properties.