The box product and Dugald's early work on distance transitive graphs

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From permutation groups to model theory:
a workshop inspired by the interests of Dugald Macpherson,
on the occasion of his 60th birthday

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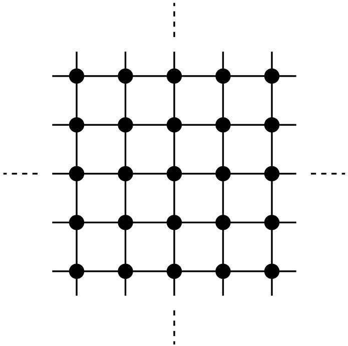
Example 1. $\mathrm{Aut}\,(\mathbb{Z}^2)$ is not distance transitive



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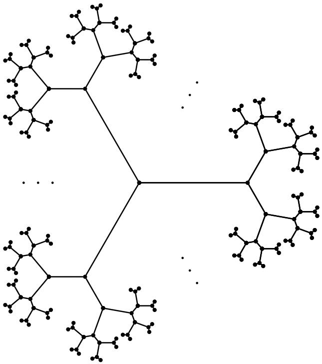
Example 2. $\mathrm{Aut}\,(T_3)$ is distance transitive



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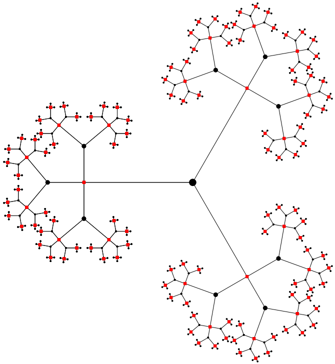

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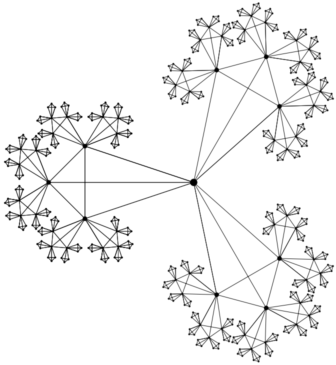

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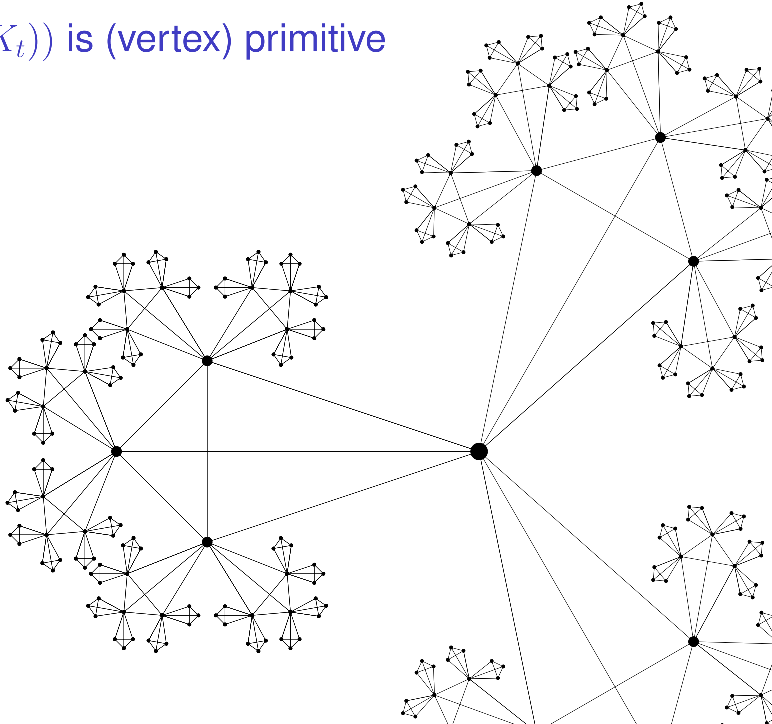
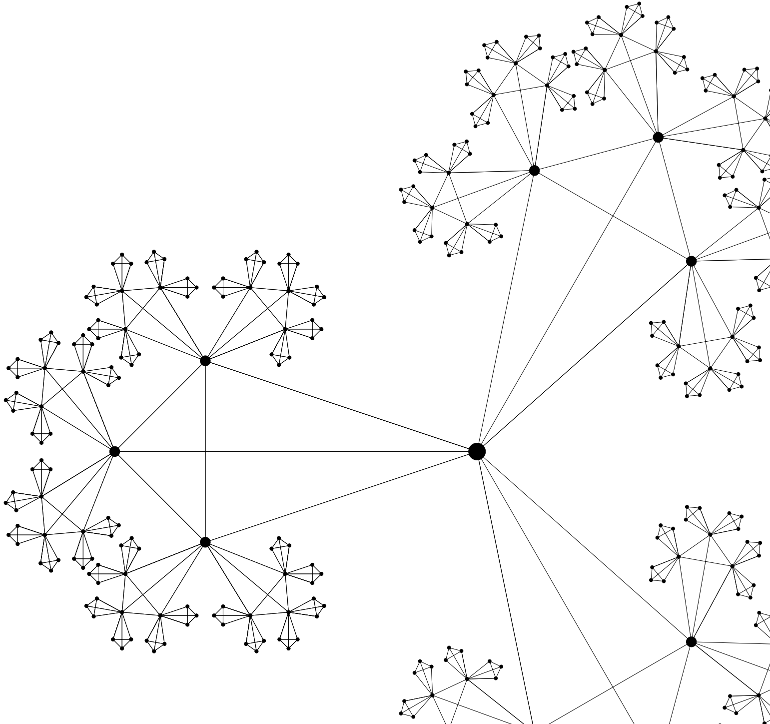
This proved a conjecture of Chris Godsil. The same result was also obtained independently by A. A. Ivanov in '83.

$\mathrm{Aut}\,(\Gamma(s, K_t))$ is (vertex) primitive





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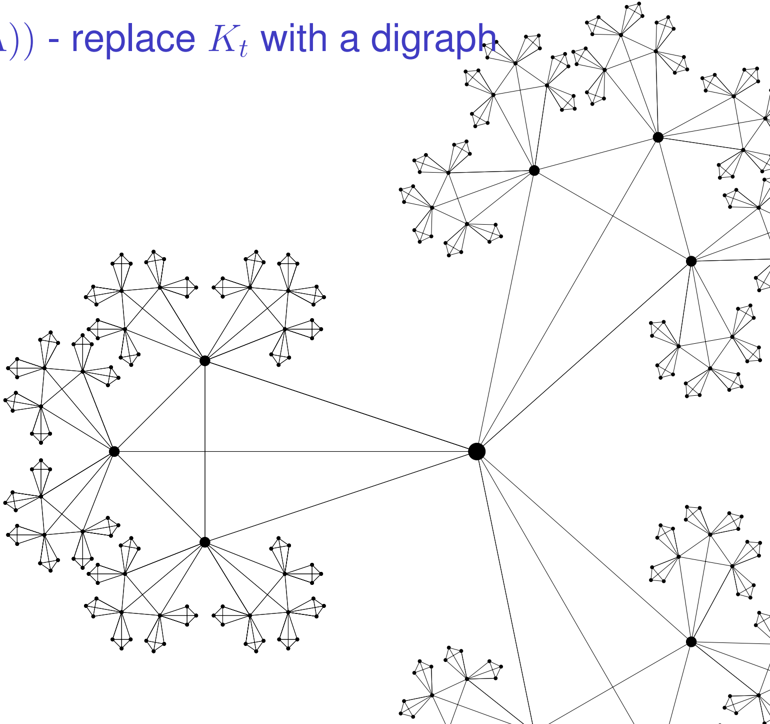

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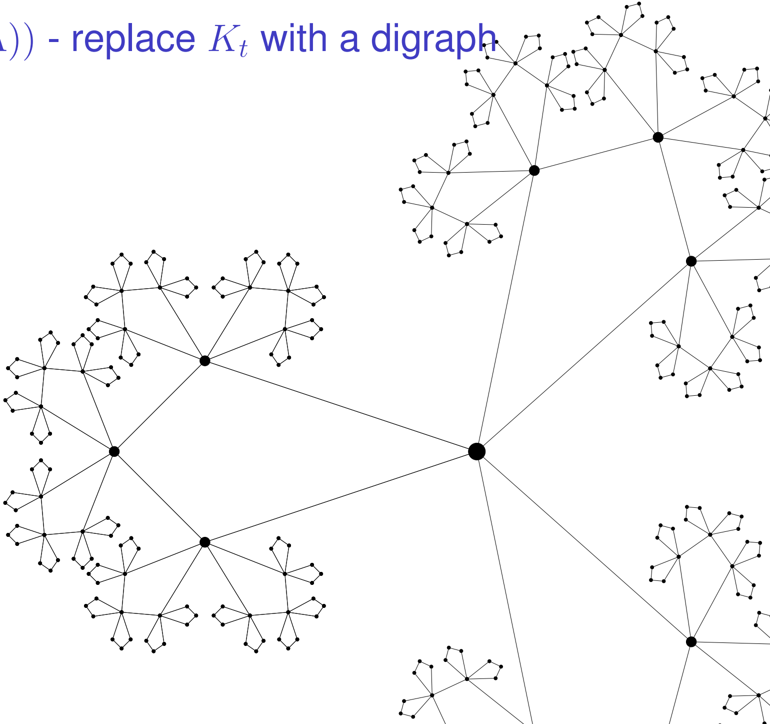

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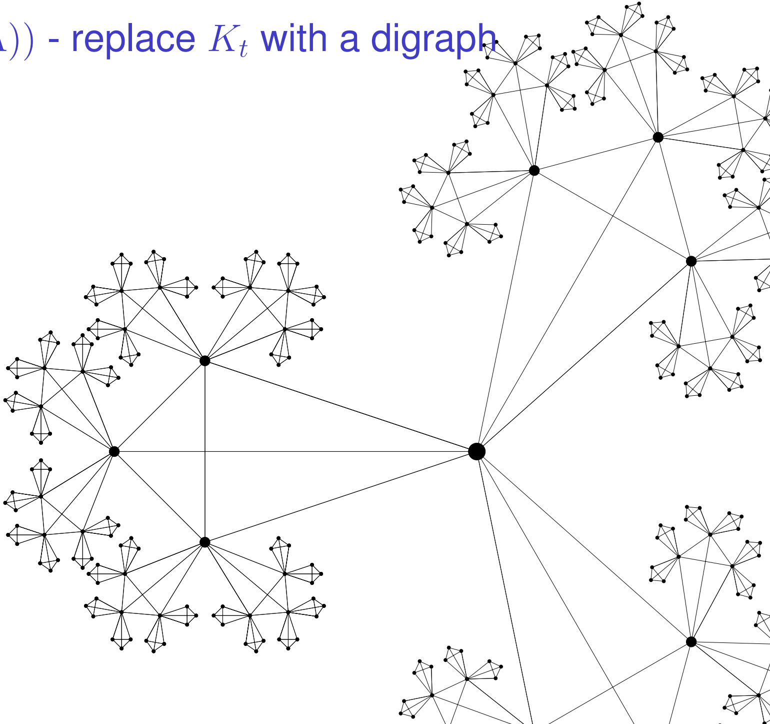

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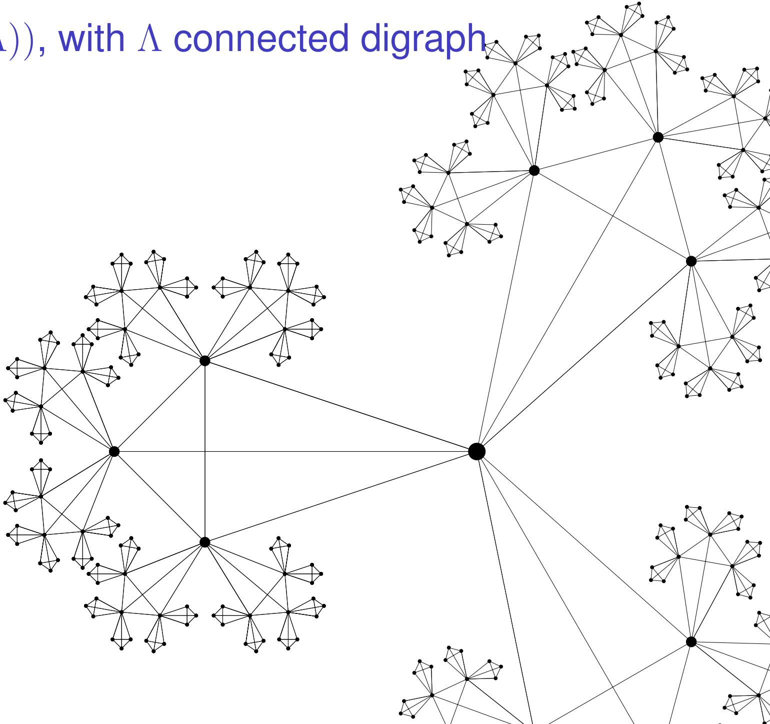

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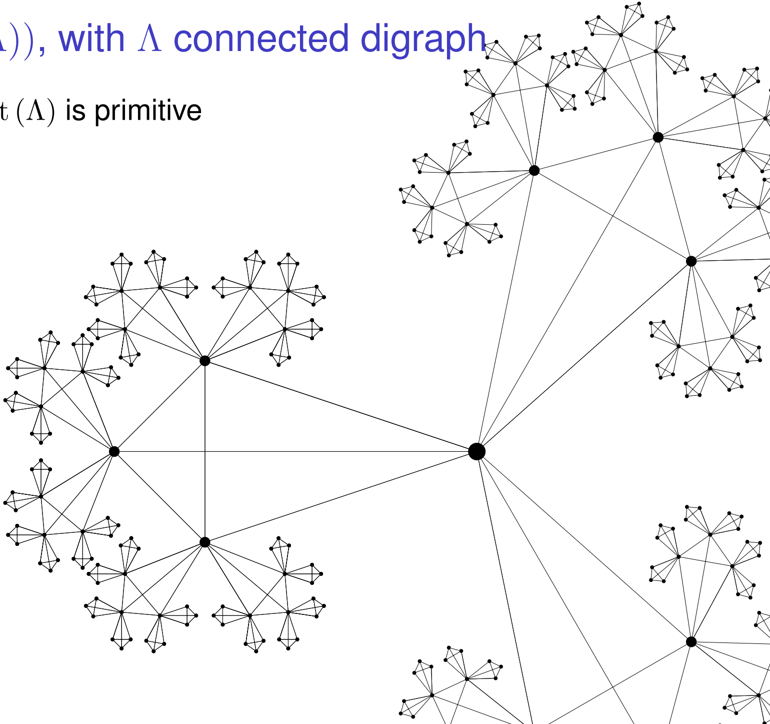
- If $\mathrm{Aut}\,(\Lambda)$ is primitive



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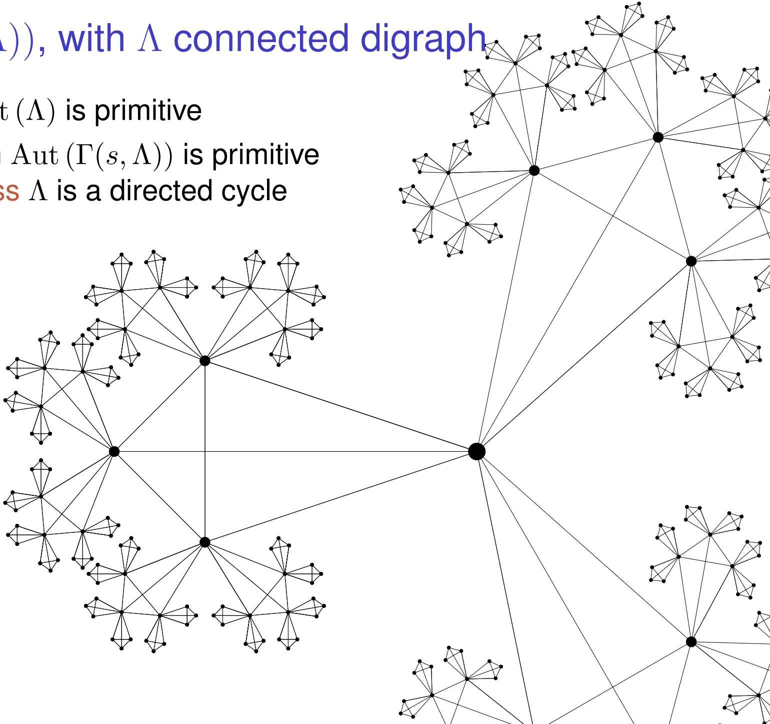
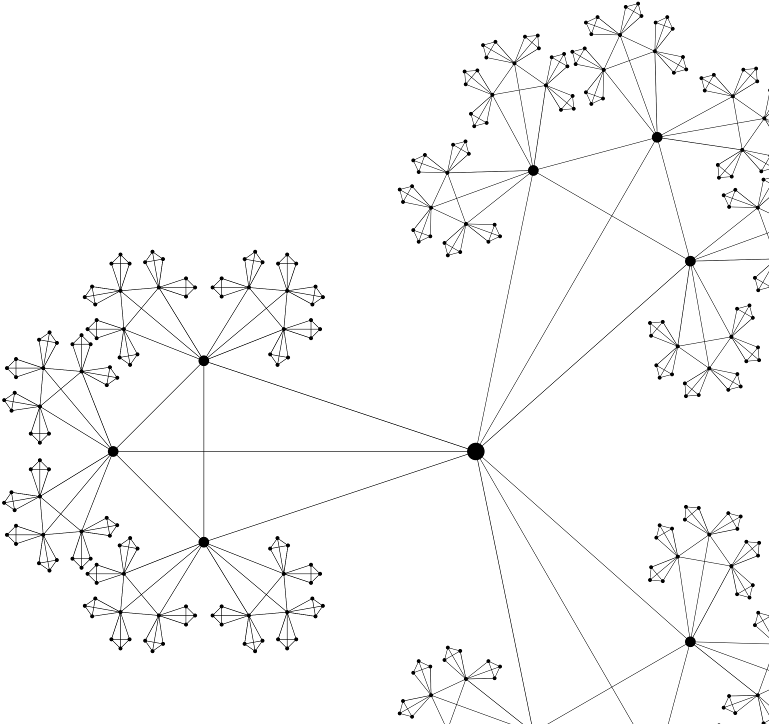
- If $\mathrm{Aut}\,(\Lambda)$ is primitive
- Then $\mathrm{Aut}\,(\Gamma(s,\Lambda))$ is primitive unless Λ is a directed cycle



The box product

$M \boxtimes N$



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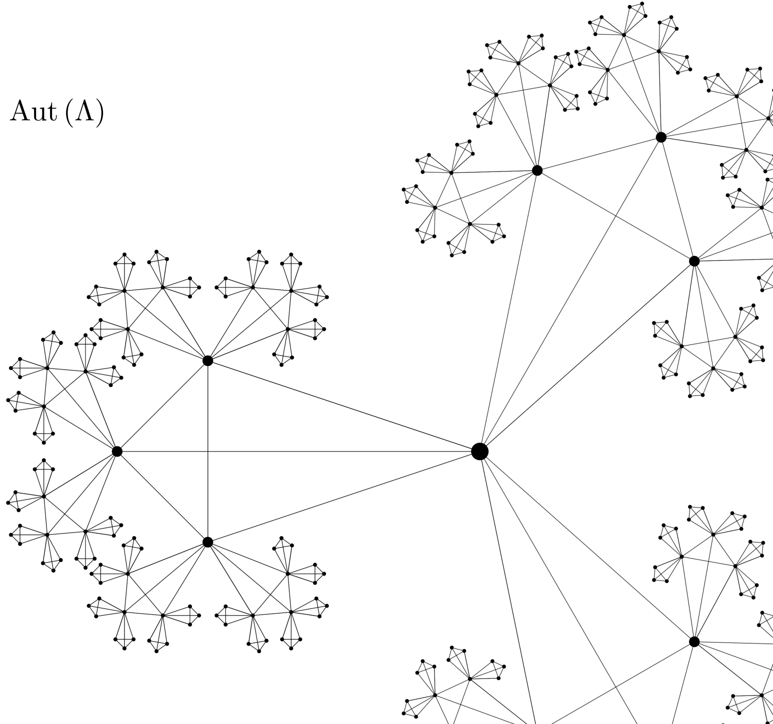
- $M \leq \mathrm{Aut}\,(\Lambda)$



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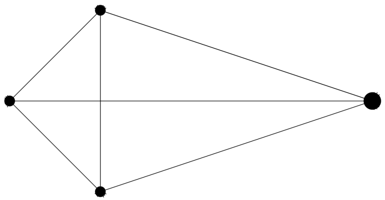

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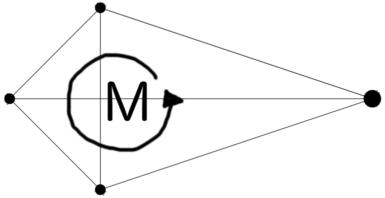

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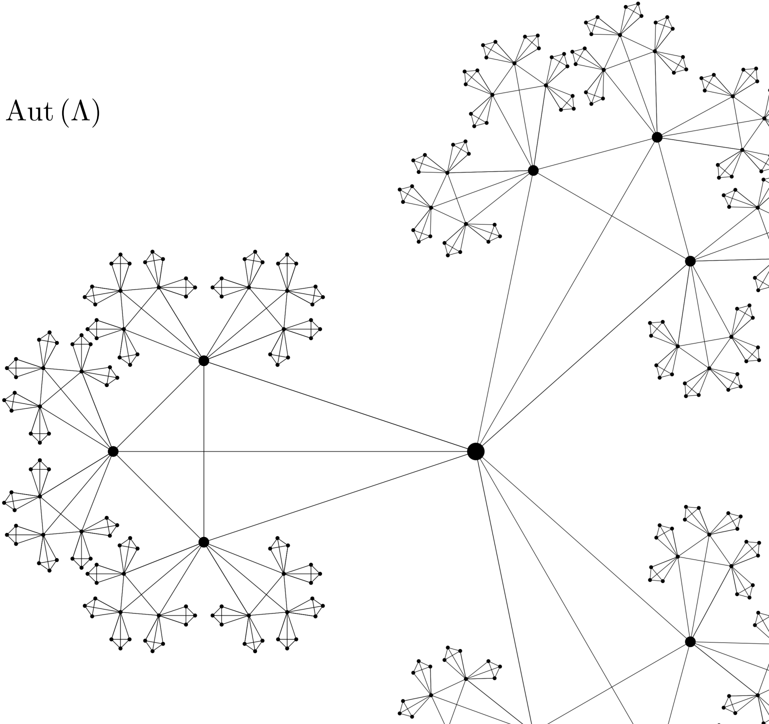

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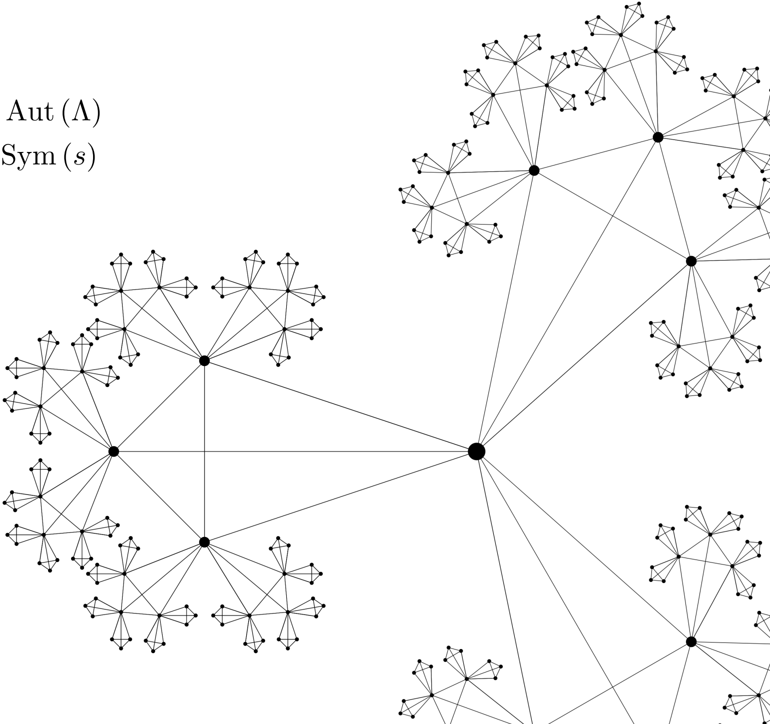

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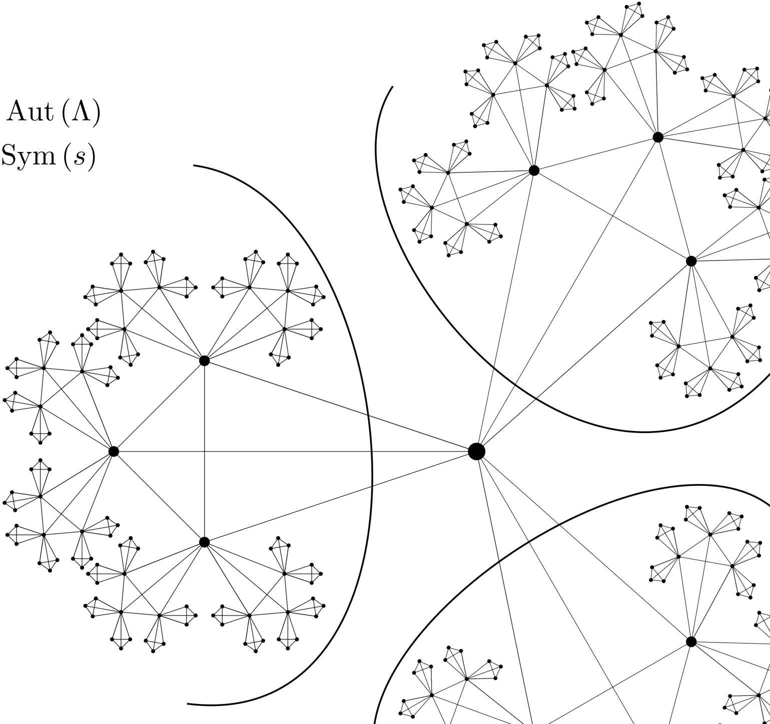

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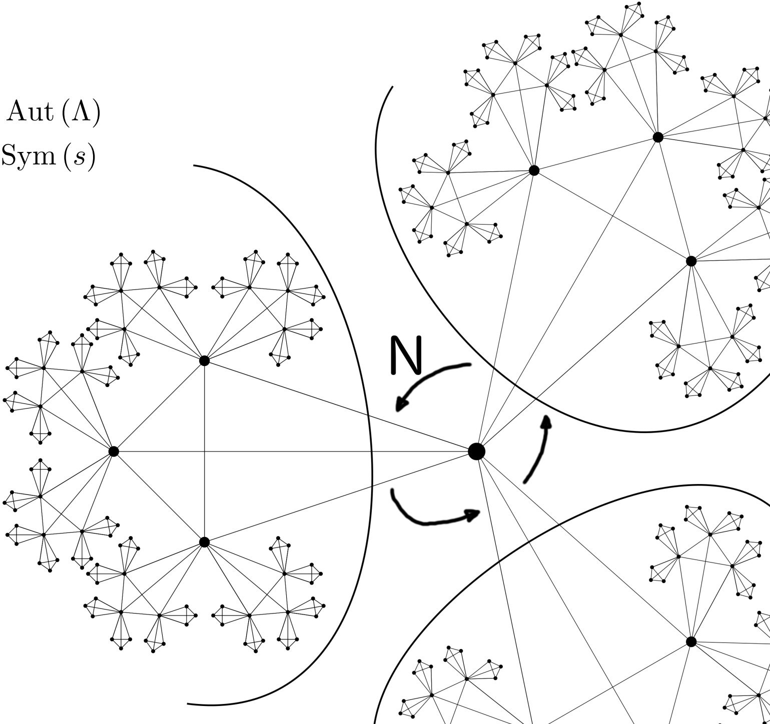

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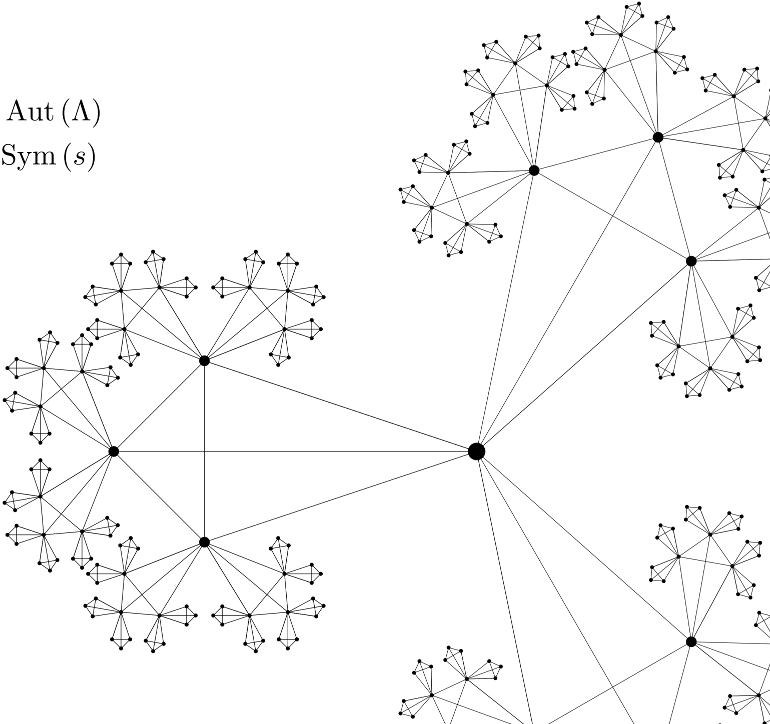

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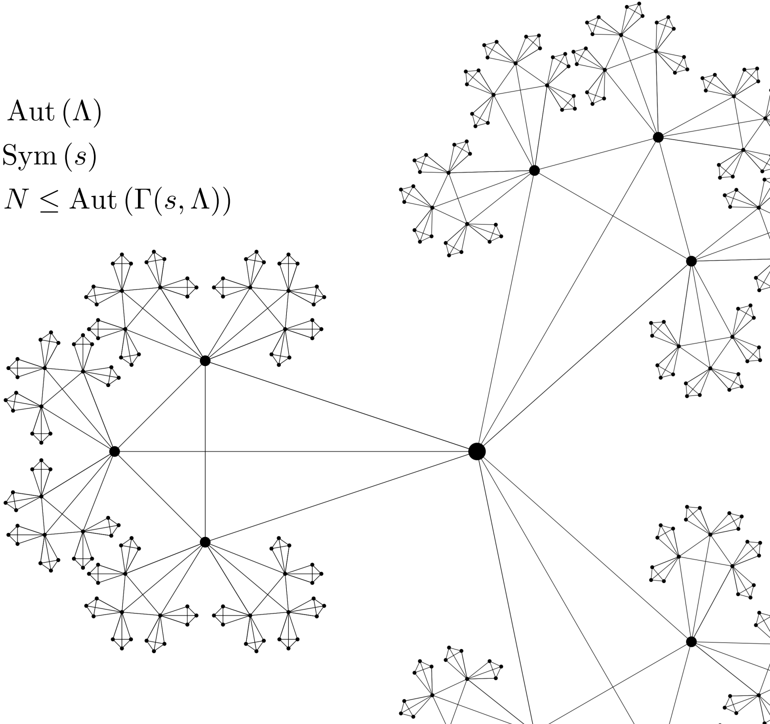

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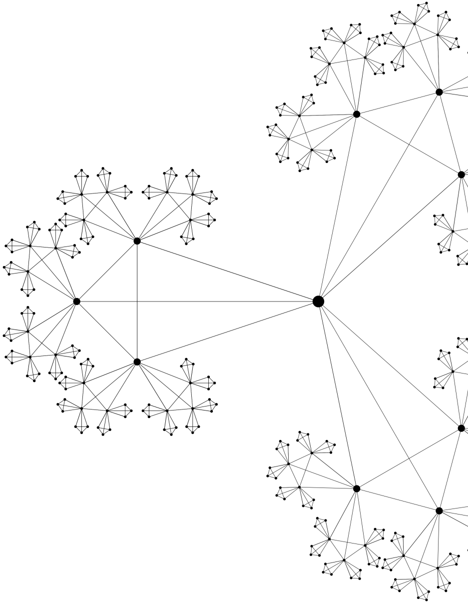

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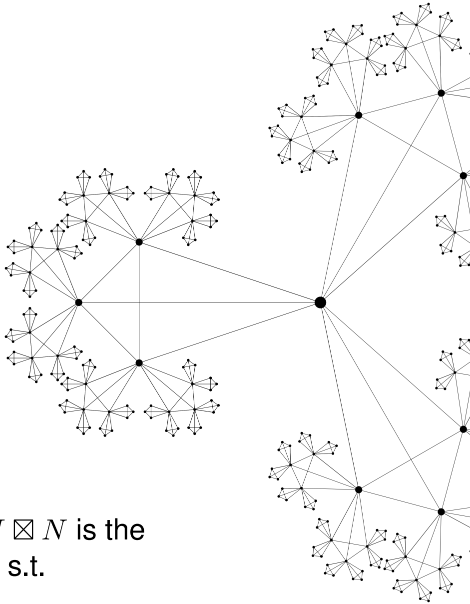

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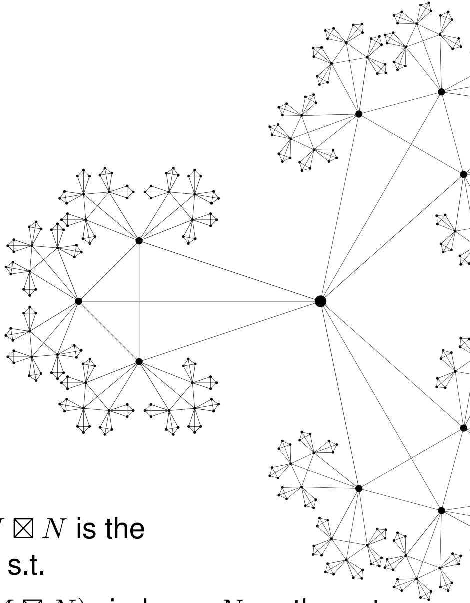

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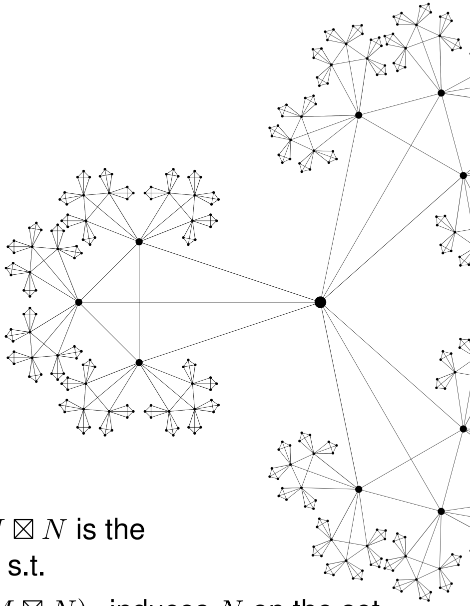

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- $\forall$ lobes Λ', the stabiliser $(M \boxtimes N)_{\{\Lambda'\}}$ induces M on Λ'.

Why is ⊠ interesting?

It behaves like the wreath product . . .

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Theorem (poss. attributable to W. Manning, early 20th C)
$M \operatorname{Wr} N$ is primitive (in product action) $\iff$

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Theorem (S., 2017)

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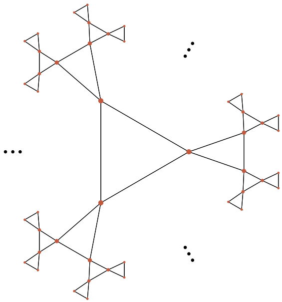
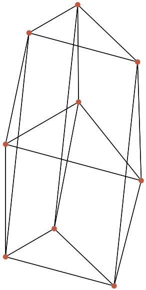
One can see the “shape” of a permutation group by looking at an orbital graph



$\mathrm{Sym}(3) \boxtimes \mathrm{Sym}(2)$



$\mathrm{Sym}(3)\,\mathrm{Wr}\,\mathrm{Sym}(2)$

It can be used to easily solve a well-known problem from topological group theory

Question (Folklore) Let $\mathcal{S}$ be the class of (non-discrete) compactly generated & simple tdlc groups.

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Theorem (S., ’17) There are $2^{\aleph_0}$ pairwise nonisomorphic groups of the form $Q \boxtimes \mathrm{Sym}(3)$, where Q is an Ol’shanskiĭ group, and all such groups lie in $\mathcal{S}$.

Hence $|\mathcal{S}| = 2^{\aleph_0}$.

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Moreover, G induces on each lobe or fibre a group that is dense in G_0

All this ties in with current work with Dugald & Cheryl on transitive extensions, but that’s another talk . . .

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Thank you

Papers:

H. D. Macpherson, *Infinite distance transitive graphs of finite valency*, Combinatorica 2 (1982)

S. M. Smith, *A product for permutation groups and topological groups*, Duke Math. J. (2017)

Preprint on subdegree-finite permutation groups coming soon