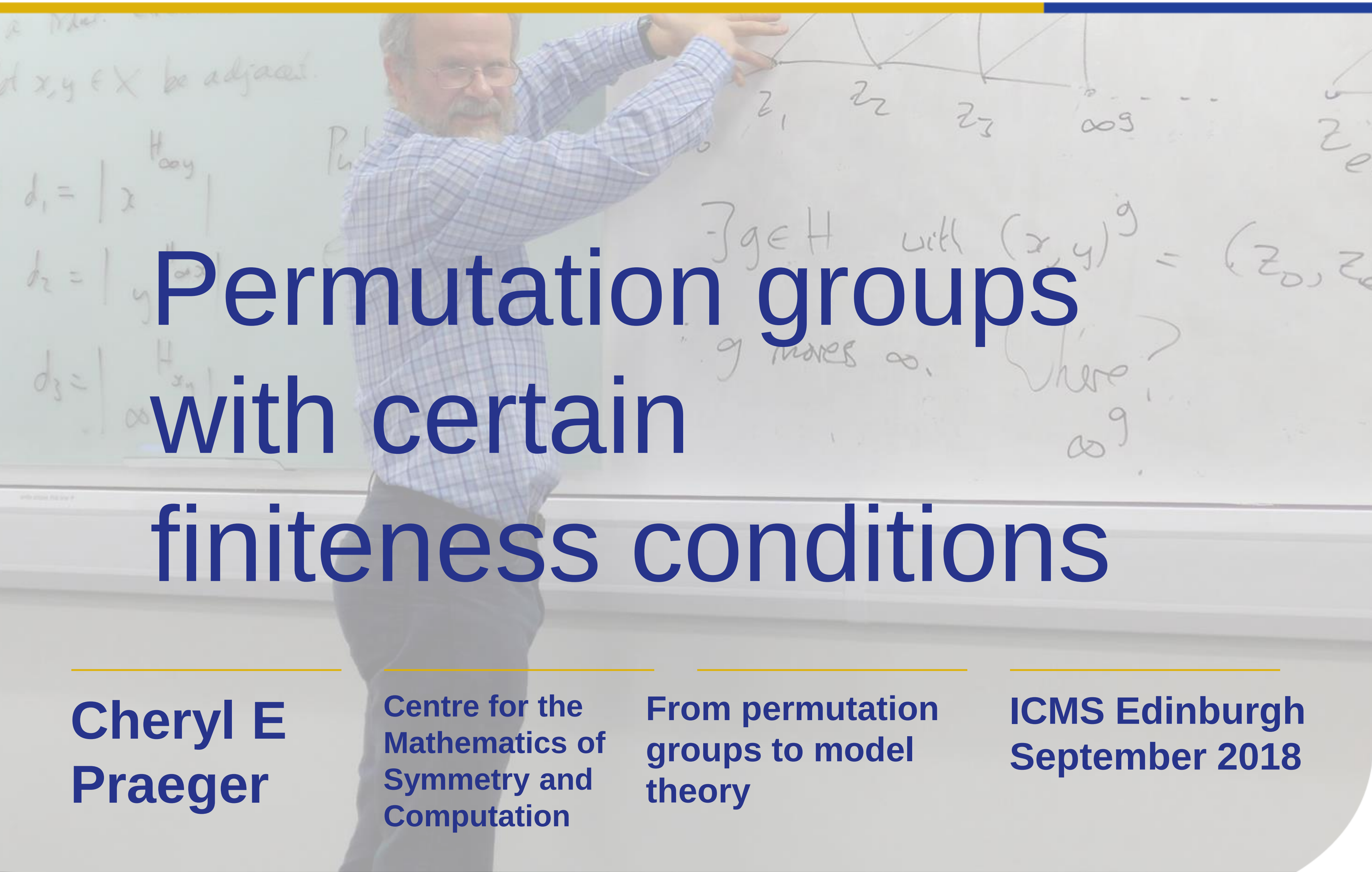

Permutation groups with certain finiteness conditions

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From permutation groups to model theory

ICMS Edinburgh September 2018

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My contributions to infinite permutation group theory

Only through collaborations with great colleagues like Dugald, Peter Neumann, Simon Smith

This lecture looks at some work with Dugald: thanks to Dugald!



First joint work

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1990 Two linked papers of Dugald's in J London Math Soc

- Up to then: knowledge about normal subgroups of Sym(X)
- 1966 Ball: Stab(Y), Y finite, maximal in Sym(X)
- 1967 Richman:
  - AlmostStab(P), partition P, maximal in Sym(X)
  - Stabiliser of ultrafilter maximal in Sym(X)
- D + Peter Neumann initiates study of various types of (non-normal) subgroups of Sym(X)
  - Highly influential – sufficient conditions for proper subgroup G to lie in maximal
- D + CEP Studies maximal subgroups
  - For X countably infinite, each "non-highly transitive" G < Sym(X) contained in maximal subgroup

G highly transitive on X means: pointwise stab of Y transitive on X-Y, for all finite Y

Finite Case

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Study of maximal subgroups of Sym(X) closely linked with:

- O'Nan-Scott Theorem: Rough structural description of finite primitive groups
  - Original formulations – described maximal subgroups of Sym(X), X finite
  - Later versions subdivided all primitive groups according to "structure"

- Dugald's 1990 papers and others: showed same could NOT be true for describing ALL infinite primitive groups

- Question: Could some version give useful description of some classes of primitive groups?

Rough version: O'Nan-Scott Theorem

Maximal subgroups H of Sym(X) (X finite) satisfy one of:

- Maximal intransitive subgroups: Subset stabilisers $\mathrm{H} = Stab(Y),\ Y \subseteq X$

Among transitive subgroups,

- Maximal imprimitive subgroups: Partition stabilisers $\mathrm{H} = Stab(\ P\ ),$ for some nontrivial partition P of X

Among primitive subgroups,

- Maximal affine subgroups: Stabilisers of affine structures $\mathrm{H} = AGL(\ V\ )$ with X identified as a vector space V

- Maximal product type subgroups: Stabilisers of product structures $\mathrm{H} = Sym(\ Y\ ) \wr Sym(\ell)$ with X identified as a Cartesian product Y^{ℓ}

- Or H is Simple Diagonal group, or Almost Simple group

Second joint work

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1994 Infinitary versions of the O'Nan-Scott Theorem (Proc LMS)

View Sym(X), X infinite, as topological group

- Basis of open neighbourhoods of identity:
  - Stab(Y), pointwise stabilisers of finite subsets Y
- G < Sym(X) closed if and only if G = Aut(1^{st} order structure)
- Imposed conditions on certain closed subgroups
- That is, considered primitive permutation groups with these conditions

Second joint work with Dugald

1994 Infinitary versions of the O'Nan-Scott Theorem (Proc LMS)

Conditions: G < Sym(X)   X infinite

- G primitive on X
- G has minimal closed normal subgroup M
- M has minimal closed normal subgroup K

Minimal subject to being closed;
Do not always exist

Obtain possible structures for G:

- Affine: $K = C_p$ and M translation group
- M is nonabelian and regular on X
- Almost topologically simple [M has no proper nontrivial closed normal sgp]
- Diagonal
- Invariant Product structure

Second joint work with Dugald

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- Suggests: Useful description for certain families of infinite primitive groups possible
- But we don't have a powerful simple group classification to exploit

Second joint work with Dugald

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- Diagonal
- Invariant Product structure

- We noted: can drop adjective "closed"
- But we were concerned the conditions would become too stringent

Finite Stabilisers

2015 Simon Smith: Primitive groups with finite stabilisers (J Alg)

Conditions: G < Sym(X)   X infinite, G primitive on X, G_x finite

Simon Proved:

- G has minimal normal subgroup M – and M unique and nonabelian
- M has minimal normal subgroup K
  - K nonabelian simple and $M \cong K \times \cdots \times K = K^k$

Such M called Finitely Completely Reducible

Finite Stabilisers

2015 Simon Smith: Primitive groups with finite stabilisers (J Alg)

Conditions: $G < Sym(X)$ X infinite, G primitive on X, G_x finite

Then:

- G has unique minimal normal subgroup M
- $M \cong K \times \cdots \times K = K^k$ with K nonabelian simple

Obtain possible structures for G:

- ~~Affine: $K = C_p$ and M translation group~~
- M is nonabelian and regular on X
- Almost simple [M=K simple and not regular on X]
- ~~Diagonal~~
- Invariant Product structure - $G \leq H \wr Sym(k)$ on Y^k with $K \leq H \leq Sym(Y)$

Finite Completely Reducible transitive minimal normal subgroup

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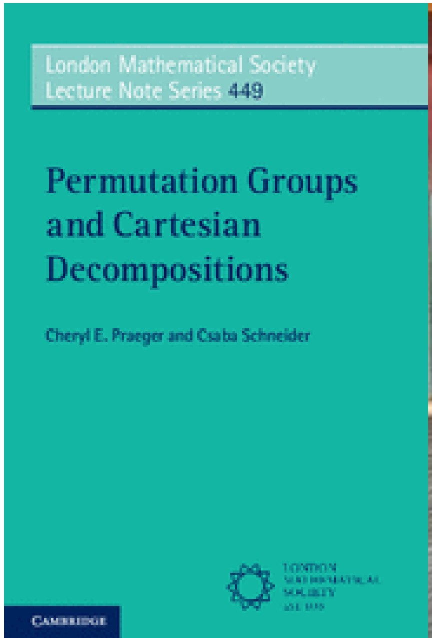
2018 Csaba Schneider & CEP: Cartesian decompositions

Simon suggested we extend to infinite permutation groups G on X

- **transitive FCR minimal normal subgroup M**

- G is innately transitive – not necessarily primitive
- $M \cong K \times \cdots \times K = K^k$ with K nonabelian simple

- Invariant product structure $X = Y^{\ell}$
- k and ℓ not necessarily equal



Finite Completely Reducible transitive minimal normal subgroup

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2018 Csaba Schneider & CEP: Cartesian decompositions

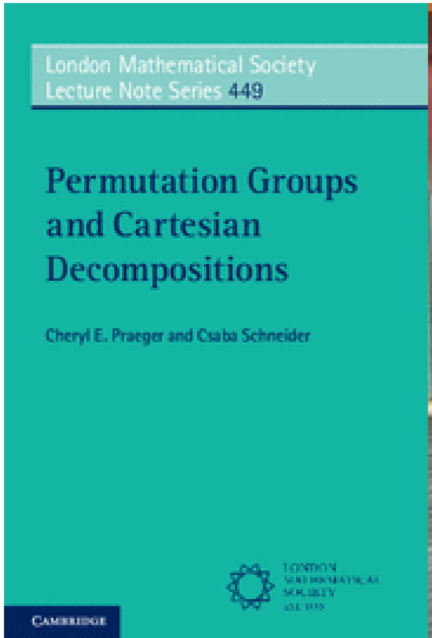
One-to-one correspondence between G-invariant Cartesian decompositions (product structures) on X and certain G_x-invariant subsets of subgroups of M called "Cartesian factorisations"

Fix $x \in X$

Cartesian Factorisation of M is a set of subgroups $\{ N_i \mid i = 1, \dots, \ell \}$ such that

1. $M_x = N_1 \cap \cdots \cap N_\ell$ and
2. for each i, $M = N_i L_i$ where $L_i = \cap \{ N_j \mid j \neq i \}$

Infinite case vastly different



Finite Completely Reducible transitive minimal normal subgroup

2018 Csaba Schneider & CEP: Cartesian decompositions

Easiest example: let $M = K_1 \times K_2$ **and** $\{ N_1, N_2 \}$ **with**

$N_1 = \{ (a,a) | \; a \in K\}$ **and** $N_2 = \{ (a, a\varphi) | \; a \in K\}$ **where** $\varphi \in Aut(K)$

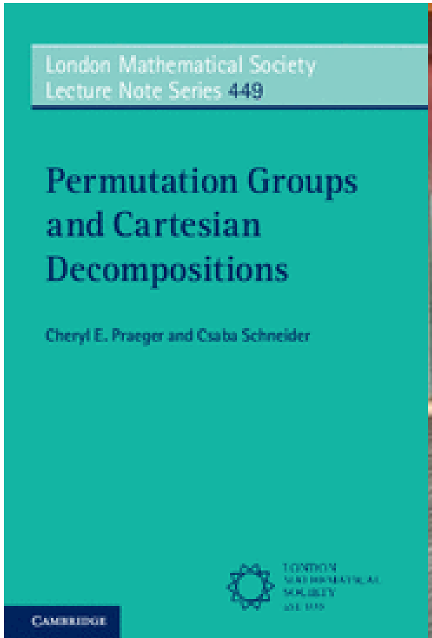
Then $\{ N_1, N_2 \}$ **is a Cartesian factorization iff**

- $M_x = \{ (a,a) | \; a \in C_K(\varphi)\}$ **and**
- $M = N_1 N_2$ **and this holds iff**

Fix $x \in X$

The map $a \mapsto a^{-1}(a\varphi)$ is surjective

φ is a uniform automorphism of K



Finite Completely Reducible transitive minimal normal subgroup

2018 Csaba Schneider & CEP: Cartesian decompositions

K finite and $\varphi \in Aut(K)$ **uniform implies K soluble**

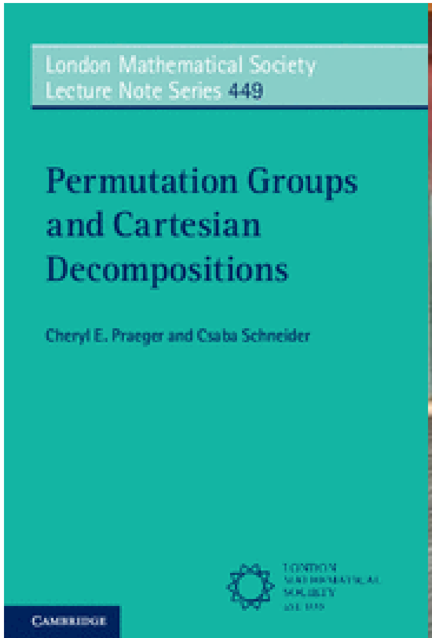
Example: $\boldsymbol{\varphi}$: $a \mapsto a^{-1}$ **if K abelian of odd order**

K = SL(2, F) with F algebraically closed characteristic p

Example: $\boldsymbol{\varphi}$: $(a_{ij}) \mapsto (a_{ij}^{p})$ **is uniform**

The map $a \mapsto a^{-1}(a\varphi)$ is surjective

φ is a uniform automorphism of K



Finite Completely Reducible transitive minimal normal subgroup

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2018 Csaba Schneider & CEP: Cartesian decompositions

For more exotic examples we formulated a problem: does there exist a group K with uniform automorphisms φ, ψ such that the map

$$a \mapsto (a^{-1}(a\varphi),\ a^{-1}(a\psi))$$

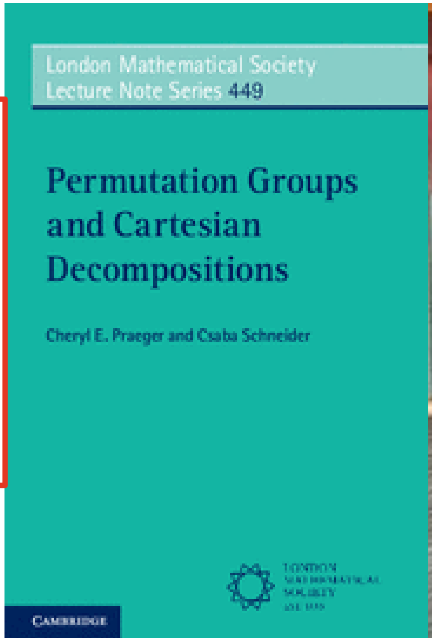
is surjective?

For such K, φ, ψ

$$M = K^6 = N_1 N_2$$

where $N_1 = \{ (a,a,a,b,b,b) \mid a,b \in K \} \cong K^2$

and $N_2 = \{ (a,b,c,a,b\varphi,c\psi) \mid a,b,c \in K \} \cong K^3$



Infinite normal subgroups transitive

2017 Peter Neumann & CEP & Simon Smith:

Considered $G \leq Sym(X)$ such that all infinite normal subgroups transitive; and G_x satisfies min-n

Min-n: Every subset of normal subgroups has a minimal element

Dichotomy:

There exists maximal finite normal subgroup or not

If G has (unique) maximal finite normal subgroup K, then

- K is semiregular (free) on X
- Quotient action of G/K quasiprimitive
- Good description – many examples

Infinite normal subgroups transitive

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Min-n: Every subset of normal subgroups has a minimal element

Dichotomy:

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Other case: very interesting

- K := < all finite normal subgroups>
- Abelian p - group, regular on X
- $K = C \times \cdots \times C \cong C^{\ell}$ where
- $C \cong \{ z \mid z^{p^k} = 1\}$ Prüfer p-group
- G/K finite and "p-adic irreducible" on K

Which finite groups G/K can arise?

Finish with another paper with Dugald

1995 Cycle types in infinite permutation groups (J Algebra)

Context: In finite primitive groups G the possible cycle types very restricted [1890s to 1930s Margraff, Manning; then 1991 Liebeck & Saxl using FSGC]

Infinite primitive groups G:
1959 Wielandt: If G contains element of finite support then $Alt(X) \subseteq G$

1995 Take this one step further:
We assumed that G contains element g with

- one infinite cycle
- finitely many, but at least one, finite cycle of length > 1
- k fixed points

Finish with another paper with Dugald

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1995 Cycle types in infinite permutation groups (J Algebra)

G contains element g with

- one infinite cycle
- finitely many, but at least one, finite cycle of length > 1
- k fixed points

Proofs use classification of infinite Jordan groups [Adeleke & Macpherson]

Then

- If k finite and G is transitive, then G is (k+1)-transitive on X
- If k is infinite and G is primitive then G is highly transitive

What can be said about other cycle types?

Finish with another paper with Dugald

1995 Cycle types in infinite permutation groups (J Algebra)

The 1995 paper + papers of John Truss:

- give some cycle types which ARE present in various infinite (non-highly transitive) primitive groups
- E.g. finite number of infinite cycles + infinitely many fixed points

1995 paper: three conjectures about kinds of primitive groups containing certain cycle structures.

- Strongest is: if G is primitive, but not highly transitive, and contains
    - Cycle type: finitely many nontrivial cycles & infinitely many fixed points
    - Then we conjecture: G contained in (not highly transitive) Jordan group

What can be said about other cycle types?

Finish with another paper with Dugald

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