Oligomorphic groups and their orbit algebras

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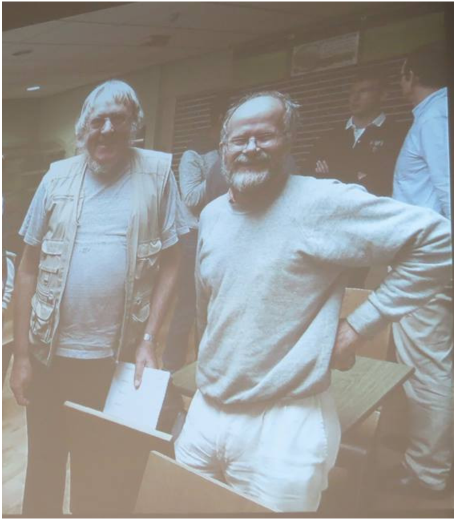
From permutation groups to model theory
Edinburgh, September 2018

Happy birthday Dugald!

$$\mathrm{Age(HDM)} = |A_5|$$

On the big screen



Durham 2015

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I will explain these things, why I was interested, why I thought it would be bad news, and what has happened since.

Definitions

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The group G may be finite or infinite; thus every finite group is oligomorphic.

The origin of the term "oligomorphic" will be explained when I discuss Fraïssé's Theorem.

Countably categorical structures

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The classic example is Cantor's theorem, asserting that the ordered set $\mathbb{Q}$ is the unique countable dense totally ordered set without endpoints. (All hypotheses except "countable" are first-order.)

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So, in this case, first-order axiomatisability is equivalent to a large amount of symmetry.

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This is the origin of the term "oligomorphic". I understand that it is used also in computer science for computer viruses which can exist in only a few different forms, as opposed to polymorphic viruses which occur in many forms.

Ages

The age of a relational structure M is the class $\mathrm{Age}(M)$ of all finite structures embeddable into M. Clearly, if M is homogeneous and $\mathrm{Age}(M)$ contains only finitely many n-element structures for all natural numbers n, then $\mathrm{Aut}(M)$ is oligomorphic.

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Fraïssé's Theorem tells us exactly when this happens …

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If these conditions hold, then M is unique (it is called the Fraïssé limit of $\mathcal{C}$).

(For the experts: I assume there is only one kind of empty set; so the amalgamation property implies the joint embedding property.)

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- G is closed in the symmetric group $\mathrm{Sym}(\Omega)$ (in the topology of pointwise convergence).

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We have $F_n^* = \sum_{k=1}^{n} S(n,k)F_k$, so (F_n) and (F_n^*) determine each other by Stirling inversion.
Also $f_n \leq F_n \leq n! f_n$, so (at least for rapid growth) (f_n) and (F_n) are not too far apart.

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I will speak mainly about the sequence (f_n).

Examples: constant

Theorem

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The first two types have $F_n = n!$ or $n!/2$ for $n \geq 2$; the third and fourth have $F_n = (n-1)!$ or $(n-1)!/2$, for $n \geq 3$. The last has $F_n = 1$ for all n.

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Further examples with f_n ultimately constant can be obtained from these by adding a finite set fixed by G.

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The group $G = S \times \cdots \times S$ (preserving all parts of a partition into k infinite parts) has $f_n = \binom{n+k-1}{k-1}$. If we allow the parts to be permuted, then $G = S \operatorname{Wr} S_k$ and f_n is the number of partitions of n into at most k parts. The same value is obtained for $G = S_k \operatorname{Wr} S$ (fixing a partition into infinitely many parts of size k).

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In all these examples, (f_n) grows as a polynomial of degree $k-1$ in n.
So all integral degrees of polynomial growth occur.

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It is not known whether other fractional exponential growth is possible. But we can have growth faster than any $\exp n^c$ for $c < 1$ but slower than exponential: $S \operatorname{Wr} S \operatorname{Wr} S$ is an example.

Primitive groups

Dugald's great theorem says:

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Problem

Show that $c = 2$ is the correct value.

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See Pierre Simon's talk on Thursday for more about this.

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There is no upper bound to the growth: just take the Fraïssé limit of the class of structures containing a_n n-ary relations which hold only if all arguments are distinct, for all n, to get a sequence growing faster than (a_n). (However, for homogeneous structures over finite relational languages, f_n is bounded by the exponential of a polynomial in n.)

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I now turn to some more algebraic tools which allow us to prove some results of this form. But the final story is not yet told ...

Generating functions

We can express the counting sequences as formal power series,

$$f(x) = \sum f_n x^n, \quad F(x) = \sum F_n x^n / n!, F^*(x) = \sum F_n^* x^n / n!.$$

(The exponential generating functions for the second and third are related to the fact that they count labelled structures in the age.)

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(The exponential generating functions for the second and third are related to the fact that they count labelled structures in the age.)
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The orbit algebra

We can bring in more structure. First we define the "reduced incidence algebra" of finite subsets of Ω. Let $\mathbb{F}$ be a field of characteristic zero. Let V_n denote the vector space of functions from the set of n-element subsets of Ω to $\mathbb{F}$, with pointwise operations. We take A to be the direct sum of these spaces for all $n \geq 0$, with multiplication defined as follows: for $f \in V_n$, $g \in V_m$, let fg be the function in V_{n+m} defined by

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We also see that, if $G = S$, then $A(G)$ is the polynomial algebra in one variable generated by e.

Examples

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In particular this holds for the automorphism group of the random graph, where the generators correspond to the finite connected graphs.

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These conjectures have implications for smoothness of growth.
The first, for example, implies

$$f_{m+n}(G) \geq f_m(G) + f_n(G) - 1.$$

A proof

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Pouzet's ingenious proof (which I cannot give here) works over the complex numbers, encoding orbits by sequences and using ideas from language theory. The crucial result (which works in the algebra A without any group) asserts that, if $f \in V_m$, $g \in V_n$, and $fg = 0$, then the union of the supports of f and g is bounded by a function of m and n.

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The second conjecture remains open.

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where the multiset I is determined by the blocks of imprimitivity of G.

This implies that $f_n(G) \sim an^k$ for some $a > 0$ and $k \in \mathbb{N}$: indeed, $f_n(G)$ is quasi-polynomial in n.

A special case

Note that the algebra of invariants of a finite permutation group is of this form; the result on $f(x)$ in that case follows from Molien's Theorem, of which the Falque–Thiéry result is a wide generalisation.